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*Sets, sinners, and salvation**

ALBERT E. MEDER, JR., *Rutgers University, New Brunswick, New Jersey.*

The sinner (mathematics teacher) is tempted. He thinks, "to do something really worthwhile is time-consuming, laborious, and difficult. Therefore, let us teach sets.'

WHEN, SEVERAL MONTHS AGO, I selected the alliterative and I hope somewhat provocative title for this talk I had confidently expected that the Report of the Commission on Mathematics would have been published and distributed and I intended to make some comments on that Report in the general context that most of you would have read it. Through no fault of the Commission or its Executive Directors, past or present, publication of that Report has been delayed; I cannot therefore assume the familiarity with it that I had anticipated. However, in a different context I shall stick to the title but express somewhat different observations.

Each of the words in the title is clearly a symbol. "Sets" is intended as a symbol for modern mathematics, for curricula that have been revised and modernized, for programs that in some way differ from the traditional algebra, trigonometry, plane and solid geometry that have constituted the curriculum for so long. I use this word because somehow or other it has become synonymous in the minds of many teachers with the modern mathematics program. Yet, as I hope to show before I am through, a program without sets can be modern; a program with sets can be antiquated. Indeed, to emphasize the fact that the introduction of sets into high school mathematics is not the essence of modernization, on more than one occasion I have talked for an hour or so on the program of curricular revision without even mentioning the word.

Nevertheless, many people think of "sets" as the criterion of modernism. Let us, therefore, use it as our symbol.

However, we should note that the voices that have been crying in the wilderness for the past three or four years, and particularly the beeps that have been coming from outer space in the last year or so telling us that something has to be done about mathematics teaching, have not at all been crying, "Sets!" They have demanded more mathematics for more pupils: in many cases, more traditional mathematics for pupils for whom it is demonstrably inappropriate; in some cases more difficult mathematics, just because it is difficult. They have confused ability to compute with mathematical understanding. They have insisted that calculus is easy, though every mathematician knows that it is highly sophisticated, and have suggested that high schools ought to teach it, sometimes adding less vociferously, "to the better students." In short, there has been a babel of confused voices demanding that something, almost anything, be done. The shibboleth of "sets" is a welcome relief.

* Address delivered at the Christmas meeting of the National Council of Teachers of Mathematics, New York, N. Y., December 29, 1958.

We turn now to our second symbolic word: "sinners." Who are the sinners? Need we ask? We are, of course, we teachers of mathematics. Surely we recognize that! We have heard it said often enough.

If we teach traditional mathematics, we are out of date. If we teach modern mathematics, whatever that is, we run the risk that colleges will not accept it (or so we are told). If we teach manipulative skills, we are teaching something meaningless. If we try to develop understanding, we are told our pupils cannot combine fractions. If a pupil likes mathematics, we are probably too easy on him; if he doesn't, we are making the subject too hard.

If he can do all his homework, we are not challenging him enough; if he can't, we are giving him useless trick problems.

If we teach logarithmic computation, we encounter an engineer on the school board who always uses a calculating machine; if we de-emphasize such computations, we hear from another engineer who wants to know why we do not demand higher standards of accuracy and pay more attention to significant figures.

Most critics of the schools and of the mathematics curriculum and of mathematics instruction are quite agreed that we are the sinners. They may indict us for different sins—even contradictory sins—but they indict us just the same. "Sinners," then, symbolizes us teachers.

"Salvation," obviously, is the way out. What is it and how do we find it? Must the sinners use sets to find salvation?

Well, yes, I do think they must, but in the symbolic sense in which I have used the words. Specifically, I do not think that we can find the way out of our difficulties without revising our high school mathematics programs. In the sense that the word "sets" symbolizes modernization in spirit and content of an outmoded curriculum and unproductive methods, sinners must use sets to find salvation.

But in another sense this statement is completely false. Obviously, in the face of the clamor to do something, the introduction of a unit on sets somewhere in the course, preferably in the ninth grade, provides a tempting answer. If such a unit is put into the course, the teacher can say to principal, superintendent, and school board: "We are up to date; these criticisms do not touch us; we teach sets!"

Now, clearly, this is no real way out; there is no salvation in following this pattern. Yet do not dismiss this possibility lightly, or feel that this contingency cannot befall you. It is a very real temptation; it is subtle; it is, like most temptations, a superficial substitute for the real thing. It requires little or no labor. It is actually and literally the most serious danger facing the program of curricular revision in our schools today.

Let me repeat, explicitly, for the matter is of fundamental importance. Most teachers of mathematics have been convinced that the proponents of curricular revision have made a case: something should be done. The general public tends to believe, largely without understanding why, that something needs to be done about the mathematics program. To do something really worthwhile is time-consuming, laborious, and difficult. Therefore, let us "teach sets." This stills the clamor that something be done; it shows that we are up to date (for what is more modern than sets?); and it lets us go about our business. If it makes anyone happier to talk about the solution set of a quadratic equation instead of the roots, we will gladly do it, provided we can still obtain the quadratic formula by incorrect manipulation and find the roots (or, rather, the elements of the solution set) by routine substitution.

I say that this sort of perversion of what the proponents of curricular revision are driving at is the greatest danger the movement has to face.

I would like to give one other answer to the question: Must sinners use sets to find salvation? before developing my last point. This time my answer is "no." It is per-

fectly possible to develop a modern curriculum in mathematics oriented to the needs of the present and the future without using sets at all. Now clearly I am using the word "sets" not as a symbol, but literally.

Just as it is a perversion of the essence of curricular revision to hold that it has been accomplished by "teaching sets," so it is a perversion to maintain that there cannot be appropriate modernization of the curriculum without teaching sets. Sets, per se, have nothing to do with the matter. The objective is to develop mathematical insight, power, and understanding; to lead pupils to think as a mathematician would think; to develop the background of knowledge and meaning that will enable a pupil to solve problems that are not exactly like the type of problem that has been solved for him in the classroom or in the textbook; to help him to deal precisely with precise ideas, and to express them correctly and preferably concisely; to deal with symbols, not like checkers on a board to be moved in accordance with predetermined and memorized rules, but as aids to thought, full of meaning and denoting ideas that can be understood; to develop the power to look at a line of mathematical symbols and interpret them, to tell what they mean. These are the things that have to be done if mathematical instruction is to meet the demands placed upon it in this day and age.

Notice that I have not talked about abstraction. I have spoken of precision of thought and meaning. Thought can be precise whether it is abstract or concrete. Notice that I have not spoken of rigor: ideas correctly understood at any level of maturity will embody that standard of logical rigor appropriate to the thinker's maturity. As he grows more mature his need for higher standards of rigor will also grow.

Now if sets, in a literal sense, are to be utilized in instruction, it must be because, through their use, it becomes possible to attain these ends better than without the language and concepts of sets. In other words, sets are a means and not an end. I myself happen to believe that the language and concepts of sets have a very real contribution to make in this regard; that through this means it is possible to make clear ideas that can be expressed only vaguely and with inherent obscurity and difficulty without the use of the language of sets: variable and function are examples. But I would be the first to concede not only the right but the obligation of a teacher who did not find this true for him to omit the use of set language in his classroom.

In other words, I believe, as Robert Rourke has pointed out in an article in THE MATHEMATICS TEACHER and elsewhere, that sets offer clarification, simplification, and unification in our teaching, and I believe that *therefore* (repeat, therefore) they should be used. But I also believe that no teacher who does not find these values in set ideas should be compelled to use them.

To sum up, then: Must sinners use sets to find salvation? Yes, in the symbolic sense that "sets" signalizes a modern orientation of the curriculum. No, in the literal sense that no program is so oriented if sets are not included. No, again, if sets are introduced only as a sop to Cerberus.

Now, finally, what is involved in the search for salvation through sets, in the symbolic sense we have defined? I want to mention just four points; no doubt others will occur to you even while I present these.

1. First and foremost, there must be a recognition of the fact that it is the spirit of modern mathematics far more than any specific content that is involved in the search for salvation through curricular revision. Meaningful mathematics, mathematics as "the study and classification of all possible patterns," mathematics in which insight is more important than reasoning, must be the goal. Students must no longer think of mathematics as a collection of tricks. Rather, they must under-

stand that by the use of an imaginative analysis of a problem they can hope to "see through it," to understand its real nature and thus work out a solution.

This concept of mathematics is neither novel nor new. It is precisely the kind of mathematics that Archimedes developed. It is precisely the kind of thinking that guided Lagrange when he moved from isolated results in the solution of algebraic equations toward a general theory; it is precisely the spirit that breathes through the work of all the great mathematicians of more recent times; it is what the nineteenth century philosopher was trying to say when he listed the characteristics of mathematical thought as Imagination, Conception, Generalization, and added that logical reasoning was only the pavement on which the chariot of the mathematician rolled.

The specific subject matter through which this kind of mathematical power can be developed is relatively unimportant. The effort to develop it is of supreme importance. It is obvious, I think, that the elimination of obsolete material of a manipulative character is one way to clear the path for this kind of teaching. So also is the elimination of overemphasis on formalism, such as is common in deductive solid geometry and to some extent in plane geometry. For other comments on specifics I refer you to the forthcoming Report of the Commission on Mathematics.

2. Second, there is involved a willingness on the part of teachers to learn more mathematics. It is perfectly clear that the kind of mathematical instruction I am talking about cannot be given by a teacher who is going through the textbook a week or a day ahead of the class. Deep background knowledge is essential. This is why the Commission says in its Report that the problem of teacher education for the new mathematics is essentially one of teaching the teacher more mathematics, not a changed methodology. Anything the teacher knows can be turned to profitable use in the effort to develop mathematical insight, power, and understanding.

I have been told by the specialists in Library Service in my University that a recent study has shown that teachers of mathematics stand next to last place among teachers in the use they make of the school library. This certainly indicates that generally speaking we have not been seeking the right objectives, but have been teaching only textbook courses. Teachers must use the library to learn more mathematics themselves, and to adapt their instruction to the individual needs of their pupils. This has not been easy to do, for the necessary books have simply not existed. Happily, one of the projects of the School Mathematics Study Group gives promise of supplying this lack.

3. Third, there must be a recognition of curricular revision as a long-range problem. Closely related to the temptation I referred to earlier to "teach sets" as a simple response to popular clamor is a feeling on the part of some that pretty soon the Commission Report will be out, and then we'll know what to do. It won't be long thereafter that we'll get revised textbooks, and then the new curriculum will be in effect. Well, it just is not that simple. Such an approach merely substitutes one dull routine for another. A new curriculum is no substitute for creative teaching. Indeed, it is the other way about. Creative teaching would be a very satisfactory substitute for a new curriculum. If we had been having creative teaching of the traditional curriculum, many of the changes now being advocated would have come about without any Commission or any other expensive project.

What is essential is that teachers move in on the modification of the existing curriculum creatively, as their knowledge and skill enables them to do so, and gradually introduce the changes recommended in the Commission Report and those suggested by others, too. Do not try to do everything at once. And, of course, remember that you are not likely to do anything in the

best possible way the first time you try it. This is why it is so important to move gradually and pragmatically. It is not fair either to your pupils or to you—nor, for that matter, to those who wrote the Report—to try everything right away before it is fully understood. Only confusion can result. Let us move deliberately, intelligently, gradually, experimentally—but, of course, perceptibly—in introducing new ideas. We will be able to do more and better the next time around.

4. Fourth, there is involved a recognition that the publication of the Report of the Commission is the beginning, not the end. Having been privileged to work on this document, I am myself quite certain that it is an extremely significant publication. But it is only printed words on paper until you, the teachers of mathematics, translate it into a living reality. The worst thing that could happen to the Report would be to be fulsomely praised as a significant document, placed on the library shelves, and ignored.

What I hope will happen is: first, that you will read it; second, that you will react to it; and third, that you will try its suggestions. People have been known to react to documents without reading them. This could happen here. The Commission says, for instance, that calculus is a college subject. You may or may not like that. It also says that it should be a freshman subject in college. Some professors will not like that. It also says that it is appropriate for high schools to teach a real course in calculus to their ablest students as an advanced placement project. This can be attacked on at least three grounds. It also says that a smattering of calculus in high school is likely to do more harm than good, but the intuitive foreshadowing of calculus concepts is desirable. How one can react emotionally to this!

My point is, please do not until you have read the argument and the discussion. After that, blow up if you want to. Most people will find themselves in agreement on some points and in disagreement on others. That is all right. But do not just stop at the level of emotional reaction. Try out some of the ideas, see what happens, and then report your findings.

Some people are likely to be disappointed because there are many options and a great deal of flexibility in the Commission's program. This, however, seems to us both necessary and desirable. It is necessary because there are great individual differences among schools, teachers, and pupils in American high schools; it is desirable because the Report is conceived of as the beginning of an enterprise to be carried on by teachers, not as the pat answer handed down by a body of experts.

Sets, sinners, and salvation. Yes, we sinners will do well to seek salvation through sets—provided we do not succumb to the temptation that sets constitute a cure-all. And since this talk has been cast in a kind of homiletic mold, let me end by quoting two scriptural texts that seem to me to sum up what I have been trying to say. I hope no one will think it irreverent or inappropriate for me to use them in this fashion. I do not mean to debase their primary meaning.

First, " . . . the letter killeth, but the spirit giveth life."

Second, "How shall we escape if we neglect so great salvation . . . ."

*Logarithmic and exponential functions —a direct approach**

C. L. SEEBECK, JR., *University of Alabama, and* P. M. HUMMEL, *University of Alabama, University, Alabama.*

A direct approach to logarithms rather than the indirect approach as an inverse of an exponential function has many advantages for teaching secondary school students.

THE USUAL DEVELOPMENT of logarithms given in high school courses has several shortcomings. The logarithmic function is defined as the inverse of an exponential function. Since the student is familiar only with the integral, or at best rational, exponents, this approach does not help him see the uses of logarithms for computational purposes. Neither does it add much to his understanding of functions. Alternate definitions in terms of convergent series, or in terms of integrals, are beyond the mathematical background of most high school students. The direct approach, as a function having certain postulated properties, can easily be followed by elementary high school students. This approach does not use limits or the concepts of continuity. It also serves as a basis for later extensions and gives the student an excellent example of the function concept early in the curriculum.

In a first course the development should be limited to the common logarithm and the theorems necessary for computation. Later on, the theory can be extended to the general base, and the change-in-base procedure established. We shall develop the theory for the general base.

Let x be a member of the set of all positive real numbers and let y be a member of the set of all real numbers. Let log be the symbol for a function which establishes a correspondence between x and y such that for each x there is one and only one y. If y is the number that corresponds to x, we write $y=\log(x)$. The logarithmic function is by definition the function that has the following properties:

(1) $\log(ab)=\log(a)+\log(b)$.
(2) There is a positive number c, called the base, for which $\log(c)=1$.
(3) If $a>1$, then $\log(a)>0$.

When several bases are being used simultaneously, or if we wish to make clear the base being used, we will change notation slightly and write $y=\log_c(x)$, read, "y equals the log of x to the base c." But when only one base is being used and it is clear what it is, the subscript is unnecessary and will not be used.

Postulate (3) is stronger than necessary, and later we will change it and note the changes in the theory. In an elementary course, however, postulate (3) as stated above is the recommended one.

On the basis of the given postulates we next proceed to the proofs of the basic theorems.

* The authors wish to acknowledge the many contributions of the Mathematics Staff at the University of Alabama to the development given here.

Theorem 1: $\log(1)=0$.

Proof: Since $a=a\cdot 1$, set $b=1$ in (1) getting

$$\log(a)=\log(a)+\log(1)$$

so that

$$0=\log(1).$$

Theorem 2: $\log(1/a)=-\log(a)$.

Proof: Since $1=a(1/a)$, set $b=1/a$ in (1) and get

$$\log(1)=\log(a)+\log(1/a).$$

But $\log(1)=0$ by theorem 1, so $\log(1/a)=-\log(a)$.

Theorem 3: $\log(a/b)=\log(a)-\log(b)$.

Proof:

$$\log(a/b)=\log\left(a\cdot\frac{1}{b}\right)=\log(a)+\log(1/b)$$

$$=\log(a)-\log(b).$$

Theorem 4: If $a>b$, then $\log(a)>\log(b)$.

Proof: If $a>b$, then $a/b>1$ and by postulate (3) $\log(a/b)>0$.

Hence, $\log(a/b)=\log(a)-\log(b)>0$, and $\log(a)>\log(b)$.

This last theorem is very important and yields numerous interesting and valuable results. It proves essentially that the logarithmic function is strictly monotonic.

Corollary 4.1. $\mathrm{Log}(a)=\log(b)$, if and only if $a=b$.

For if a and b are different, one of them is larger than the other and the theorem applies.

Corollary 4.2. If $\log(b)>\log(a)$, then $b>a$.

This follows since $b<a$ contradicts Theorem 4, and the condition $b=a$ requires $\log(a)=\log(b)$.

Corollary 4.3. The base c is greater than 1.

Corollary 4.4. If $a<1$, then $\log(a)<0$.

Thus, the logarithms of numbers between 0 and 1 are negative.

Theorem 5: If n is any rational number, then $\log(a^n)=n\log(a)$.

Proof: Case 1. First, let n be a positive integer. Then

$$\log(a^n)=\log(a\cdot a\cdot\cdots a) \text{ with } n \text{ factors}$$

$$=\log(a)+\log(a)+\cdots+\log(a)$$

to n terms

$$=n\log(a).$$

Case 2. Let $n=p/q$ where p and q are positive integers.

Set $x=a^n=a^{p/q}$. Then $x^q=a^p$ and $\log(x^q)=\log(a^p)$.

Since the theorem holds for case 1, we get

$$q\log(x)=p\log(a), \text{ and } \log(x)=\frac{p}{q}\log(a).$$

Hence,

$$\log(a^n)=n\log(a).$$

Case 3. Finally, let $n=-r$, where r is a positive rational number. Then

$$\log(a^n)=\log(1/a^r)=-\log(a^r)$$

$$=-r\log(a)=n\log(a).$$

Corollary 5.1. If r is any rational number, then $\log(c^r)=r$.

To show that properties (1), (2), and (3) determine a unique function, we need the following:

Auxiliary theorem. If a and b are any two real numbers with $a<b$, then there is a rational number r such that $a<r<b$.

Proof: Since $a<b$, $b-a>0$, and for a sufficiently large positive integer N, $b-a>1/10^N$. Now consider the set of rational numbers $0, 1/10^N, 2/10^N, 3/10^N, \cdots$, together with the negatives of these numbers. They form a set which extends infinitely far in each direction, and consecutive members differ by less than $b-a$. Consequently at least one member of the set must fall between a and b.

Theorem 6: There is a unique function, log, with properties (1), (2), and (3).

Proof: Suppose there are two such functions, log and LOG, and a positive real number a for which $\log(a)<\mathrm{LOG}(a)$. By

the preceding theorem there is a rational number r such that

$$\log(a) < r < \mathrm{LOG}(a).$$

But by corollary 5.1, $r = \log(c^r) = \mathrm{LOG}(c^r)$. Therefore we have

$$\log(a) < \log(c^r) = \mathrm{LOG}(c^r) < \mathrm{LOG}(a).$$

Corollary 4.2 now requires that $a < c^r < a$, which is impossible and refutes the supposition that two such functions exist.

In an elementary course the preceding development should perhaps be carried through using base 10 and stopping with corollary 5.1. Next, the graph of $y = \log(x)$ should be sketched. Two points, (1, 0) and (10, 1), are already known, and two more, $(10^{1/4}, 1/4)$ and $(10^{1/2}, 1/2)$, can be approximated. Theorem 4 shows the monotonic nature of the function, and a fair sketch can be made.

Students can now be shown a table of mantissas with the explanation that these are the logarithms of the numbers from 1 to 10. They are next shown how the tables can be extended to the positive numbers less than 1 and to numbers greater than 10 by means of the simple relations:

$$\log(10N) = 1 + \log(N), \tag{4}$$

and

$$\log(N/10) = \log(N) - 1,$$

which are immediate consequences of properties (1) and (2) and theorem 3.

Characteristics and mantissas can now be defined, and a rule for characteristics determined. Next, the usual computational drill material should be thoroughly covered. For although the student will make fewer algebraic mistakes when taught logarithms in this manner, there is nothing that takes the place of drill to develop computational technique.

Before proceeding to the change-in-base technique and the inverse logarithmic function we need one more fundamental result.

Theorem 7: For every real number y there is a positive real number x such that $y = \log_c(x)$.

Proof: We assume the fact that if a subset of the set of real numbers has an upper bound, then it has a least upper bound. Now let S be the set of all rational numbers such that s belongs to S if $s < y$. Consider now the set c^s. It clearly has an upper bound and therefore a least upper bound, N. We will show that $y = \log_c(N)$. For suppose $\log_c(N) < y$. Then there is a rational number s in S such that $\log_c(N) < s < y$, and therefore $N < c^s$. Thus, N would not be an upper bound for the set c^s, which is contrary to fact, so $\log_c(N)$ cannot be less than y. Next suppose that $\log_c(N) > y$. Then there is a rational number r, not in S, such that $\log_c(N) > r > y$, and therefore $N > c^r$. Since c^r is an upper bound for the set c^s, N is not the least upper bound. Since this too is contrary to fact, we must have $y = \log_c(N)$.

We can now proceed to the relationship of logarithms with different bases and develop a change-in-base procedure. To this end we next prove

Theorem 8: Let k be a positive real number. Then $f(x) = k \log_c(x)$ is a logarithmic function, $\log_{c'}(x)$, with $c' > 1$.

Proof: Since

$$\begin{aligned}(ab) &= k \log_c(ab) = k \log_c(a) + k \log_c(b) \\ &= f(a) + f(b), \text{ property (1) holds.}\end{aligned}$$

If $a > 1$, $\log_c(a) > 0$, and $f(a) = k \log_c(a) > 0$, so property (3) also holds. In view of theorem 7 there is a positive real number, call it c', such that $\log_c(c') = 1/k$. c' will be > 1 since $1/k > 0$.

Now $f(c') = k \log_c(c') = k(1/k) = 1$, and we have property (2) with c' being the base. Since the three properties uniquely determine a logarithmic function, the theorem is established.

Using the same notation as in the theorem just proved, set $x = c$, and we get $f(c) = k \log_c(c) = k$. We can now readily verify

Corollary 8.1. $k = \log_{c'}(c) = 1/\log_c(c')$.

Corollary 8.2.

$$\log_{c'}(x) = \log_{c'}(c) \cdot \log_c(x) = \frac{\log_c(x)}{\log_c(c')}.$$

In view of theorem 7 and the consequences of theorem 4, it is clear that the functional relation, $y=\log_c(x)$, establishes a one-to-one correspondence between the positive real numbers (x) and the set of all real numbers (y). We may therefore consider the inverse function.

The inverse function we shall temporarily call the c-function. Thus, by definition, we shall say that

(5) $$y=f_c(x),$$

read "y equals the c-function of x," if and only if $x=\log_c(y)$.

From the definition of the c-function it is immediately evident that

$$f_c(\log_c(y))=y, \text{ and } \log_c(f_c(x))=x.$$

It also follows at once that

(6) $f_c(0)=1$, since $0=\log_c(1)$,

(7) $f_c(1)=c$, since $1=\log_c(c)$,

(8) and for rational r,

$$f_c(r)=c^r, \text{ since } r=\log_c(c^r).$$

Other properties of the c-function are readily obtained, for if

$y_1=f_c(x_1)$, so that $x_1=\log_c(y_1)$, and

$y_2=f_c(x_2)$, so that $x_2=\log_c(y_2)$,

then

$$x_1+x_2=\log_c(y_1)+\log_c(y_2)=\log_c(y_1\cdot y_2),$$

and therefore $y_1\cdot y_2=f_c(x_1+x_2)$, which establishes

Theorem 9: $f_c(a)\cdot f_c(b)=f_c(a+b)$:

Similarly,

$$x_1-x_2=\log_c(y_1)-\log_c(y_2)=\log_c(y_1/y_2),$$

so that $y_1/y_2=f_c(x_1-x_2)$, and we have

Theorem 10: $f_c(a)/f_c(b)=f_c(a-b)$.

Corollary 10.1. $f_c(-b)=1/f_c(b)$.

Also, if n is rational,

$nx=n\log_c(y)=\log_c(y^n)$, so that

$y^n=f_c(nx)$, which proves

Theorem 11: $(f_c(x))^n=f_c(nx)$.

In view of the relations (6), (7), and (8) along with theorems 9, 10, and 11, we see that for rational x the c-function has precisely the properties of the exponential function. That is, for rational x,

(9) $$f_c(x)=c^x.$$

Because of the fact that the inverse logarithmic function is exponential for the rational numbers, we extend it by definition to include all real numbers. Thus by definition, if x is any real number,

$$y=c^x \text{ means } x=\log_c(y).$$

The notation $f_c(x)$ can now be replaced by the simpler exponential notation, and the results rewritten in exponential form. For example, theorem 9 becomes the familiar

$$c^a\cdot c^b=c^{a+b}.$$

Similarly for the other results.

A final advantage of this approach to the logarithmic function is encountered in the calculus. Let $F(x)=\int_1^x dt/t$. It can be shown that $F(x)$ is a logarithmic function by showing that it has the three defining properties (1), (2), and (3). That it has property (1) can be shown as follows:

$$F(ab)=\int_1^{ab} dt/t=\int_1^{a} dt/t+\int_a^{ab} dt/t.$$

Changing variables in the last integral on the right gives

$$t=ay,\ dt=a\cdot dy,$$

and

$$F(ab)=\int_1^{a} dt/dt+\int_1 dy/dy$$

$$=F(a)+F(b), \text{ which is property (1).}$$

It is clear of course that property (3) holds, and there remains to show that there is a base, which we shall call e, such that $F(e)=1$. To show the existence of such a base, and in fact to approximate its value, consider the trapezoidal areas shown in Figure 1. The one below the curve is tangent to it at the midpoint of the interval so that its area is the same as

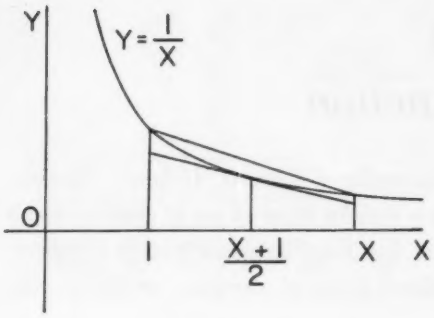

Figure 1

the rectangle with height equal to the midpoint ordinate. Since the curve lies between the two trapezoids, we have the following inequalities:

$$(10)\quad \frac{2}{x+1}(x-1) < \int_1^x dt/t < \frac{1}{2}(1+1/x)(x-1) = \frac{x^2-1}{2x}.$$

From these inequalities we find that $F(2)<1$ and $F(3)>1$, so that there is an e, $2<e<3$, for which $F(e)=1$. Since this establishes $F(x)$ as a logarithmic function we will now change to the usual notation and write $F(x)=\log_e(x)$, or the more commonly employed $\ln(x)$.

To approximate e let us set $x=1+h$ and (10) becomes

$$(11)\quad \frac{2h}{2+h} < \ln(1+h) < \frac{h(h+2)}{2(h+1)}.$$

Now by the change-in-base relation given in corollary 8.2,

$$\ln(1+h) = \frac{\log_{10}(1+h)}{\log_{10} e}.$$

Using this in (11) leads to

$$\frac{2(h+1)}{h(h+2)}\log_{10}(1+h) < \log_{10} e < \frac{2+h}{2h}\log_{10}(1+h).$$

In this last relation set $h=.001$. In a seven place table of mantissas is found $\log_{10}(1.001)=.0004341$. However, this value contains a rounding error, so we use

$$.00043405 \leqq \log_{10}(1.001) \leqq .00043415.$$

Performing the appropriate arithmetic gives

$$.434266 < \log_{10} e < .434368,$$

and

$$2.7181 < e < 2.7188.$$

We shall clarify an earlier statement concerning postulate (3). To this end we will change postulate (3) and note how the theory is affected. Let the new postulate be

(3′) If $a<1$, then $\log_c(a)>0$.

It is not difficult to follow through the preceding development and make the necessary changes. We will state the changes required by postulate (3′) without proof since they closely parallel the proofs already given.

Theorem 4′: If $a<b$, then $\log_c(a)>\log_c(b)$.

Corollary 4′.2. If $\log_c(a)>\log_c(b)$, then $b>a$.

Corollary 4′.3. The base c is less than 1.

Corollary 4′.4. If $a>1$, then $\log_c(a)<0$.

The greatest change occurs in theorem 8 as k may be either positive or negative and we may change from a base less than 1 or greater than 1 to another base of either type.

Theorem 8′: Let k be any real constant different from zero. Then the function $f(x)=k\log_c(x)$ is a logarithmic function, $\log_{c'}(x)$, and

if $k>0$ and $c>1$, then $c'>1$,

if $k<0$ and $c>1$, then $c'<1$,

if $k>0$ and $c<1$, then $c'<1$,

if $k<0$ and $c<1$, then $c'>1$.

The other theorems and corollaries remain the same. Thus we see that postulate (3) gives a monotonic increasing logarithmic function with a base greater than 1 whereas postulate (3′) gives a monotonic decreasing logarithmic function with a base less than 1.

Useful generalizations of the concept of function

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A function is a special kind of set of ordered pairs of elements, but these elements can be numbers, ordered pairs of numbers, or statements.

WHEN THE CONCEPT of function[1] is defined in a suitable manner, valuable insight may be gained into the relationships between some of the more elementary aspects of mathematics and some of those aspects generally reserved for a later point in mathematics education as being too sophisticated for the high school student.

It is well known that a function may be defined as a special kind of set of ordered pairs. (If $f(x)$ is the result of evaluating the function f for the value x, then the ordered pair $(x, f(x))$, and all such ordered pairs for values of x for which the function is defined, make up a set. This set of ordered pairs is then said to be the function f.) In this manner, the function concept can be built up from these two relatively more primitive concepts, set and ordered pair. If one does this, the set of all elements which occur as first members (values of x) of ordered pairs in the function is commonly called the *domain* of the function. Similarly, the set consisting of all second members (values of $f(x)$) of ordered pairs in the function is called the *range* of the function.

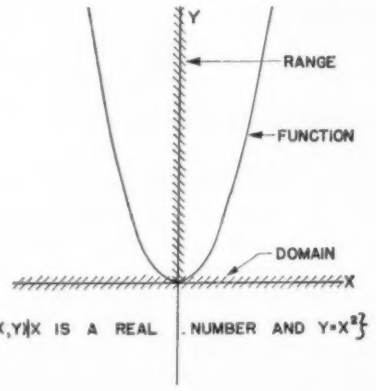
If we adopt the notation $\{x/\cdots\}$, where the three dots " . . . " represents some condition on x, to abbreviate the phrase, "the set of all elements x such that . . . ," we would write the function commonly called "x^2" as "$\{(x, y)/x$ is a real number and $y = x^2\}$." (Read: "the set of all ordered pairs (x, y) such that x is a real number and y equals x-squared.") The domain of this function is the set $\{x/x$ is a real number$\}$, while the range is the set $\{y/y$ is a real number and $y \geqq 0\}$. (See Figure 1.)

Although many people customarily think of a function as being defined by a simple rule such as

$$f(x) = x^2,$$

there is no reason to restrict the function concept in this fashion. For example, Figures 2*a* through 2*d* are graphs of perfectly respectable functions.

Figure 1



[1] In this article, the word *function* is reserved for what is sometimes called a *single-valued function*. The word *relation* is used to indicate a many-valued function.

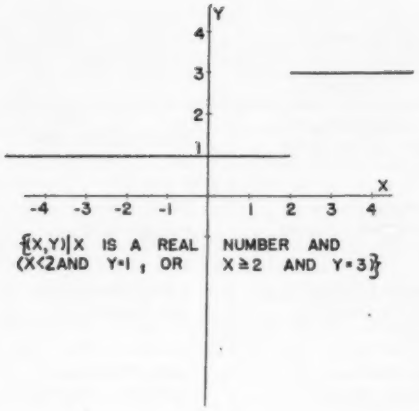


Figure 2a

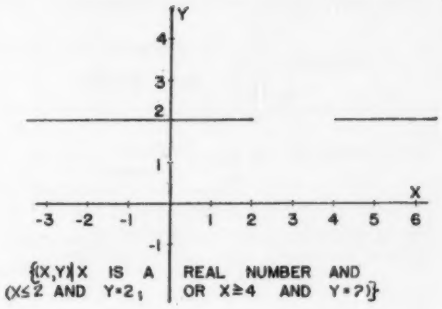


Figure 2b

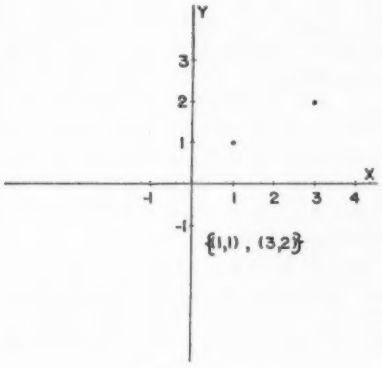


Figure 2c

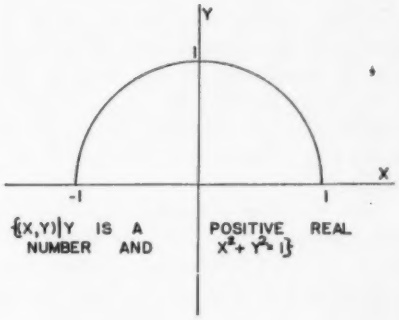


Figure 2d

Figure 2*a* indicates that we are not requiring continuity; Figure 2*b*, that the domain need not be the entire real line; Figure 2*c*, that the function may be defined by enumerating the ordered pairs which go to make up the function; and Figure 2*d* shows that the function may be defined implicitly.

Up to this point, however, we have considered only functions whose domain and range are sets of real numbers; that is, functions which transform one real number into another real number. Nothing has been said, for example, of functions of two variables. It is not difficult to see that a function of two variables may be characterized as a function whose domain is a set of ordered pairs (of numbers) and whose range is a set of numbers. The function $f(x, y) = x^2+y^2$ transforms ordered pairs (x, y) of numbers into the single number x^2+y^2; for example, (2, 3) into 13 (i.e., into 2^2+3^2); (3, 4) into 25 (i.e., into 3^2+4^2); etc. Perhaps it is a little awkward, at first, to conceptualize an ordered pair, one of whose parts is also an ordered pair, for example: ((2, 3), 13);[2] but this awkwardness disappears with a little practice.

[2] For convenience in writing, the internal set of parentheses may be replaced by a semi-colon: (2,3;13).

TABLE 1

EXAMPLES OF FUNCTIONS WITH VARIOUS DOMAINS AND RANGES

DOMAIN	RANGE		
	Set of numbers	Set of ordered pairs of numbers	Set of statements
Set of numbers	1. ordinary functions	2. parametric representation of curve	3. condition on one variable
Set of ordered pairs of numbers	4. function of two variables	5. vector function of two variables	6. condition on two variables
Set of statements	7. truth value function	8. (no obvious useful example)	9. one-place logical operator (negation)

(It may be noted in passing that since a complex number may be thought of as an ordered pair of real numbers [i.e., $a+bi$ may be considered as the ordered pair (a, b)], a real function of a single complex variable may be considered in a manner identical to a function of two [real] variables.)

Thus we have examples of a function from a set of numbers to a set of numbers (i.e., a function with a set of numbers for its domain and a set of numbers for its range) and of a function from a set of ordered pairs of numbers to a set of numbers. We ask, "Is it possible to give a function which has a range other than a set of numbers?" This question is answered in the affirmative by Table 1, which gives examples of functions with three different kinds of sets used as domain and range.

We have already discussed cells 1 and 4 of Table 1. Before considering functions involving statements, we will consider cells 2 and 5, which deal with functions whose domains are similar to those of the two kinds of functions already considered, but whose ranges are sets of ordered pairs of numbers, rather than sets of numbers. As an example of cell 2, the function

$$\{(t, (x, y))/x = \cos t, y = \sin t\}$$

is often used to describe the unit circle. If we call this function "f," we find that $f(0) = (1, 0)$, $f(\pi/4) = (\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$, $f(\pi/2) = (0, 1)$, $f(\pi) = (-1, 0)$, $f(3\pi/2) = (0, -1)$, and $f(2\pi) = (1, 0)$. Thus, as t goes from 0 to 2π, the point $f(t)$ (i.e., the ordered pair (x, y), where $x = \cos t$ and $y = \sin t$) describes a circle (in the counterclockwise direction) of unit radius about the origin as a center. Such parametric representations of a curve are an important device in beginning calculus.

Moving to cell 5, we note that a vector function of two variables may be characterized as a function from a set of ordered pairs of numbers to a set of ordered pairs of numbers. For example, the wind magnitude and direction at each point of the earth's surface may be so represented as a vector function of two variables. This description of a vector function in terms of ordered pairs can become an important tool in cutting through the fog that enshrouds this topic (vector) in the minds of many of us and of our students. This application to vectors, vector functions, and the related concept of tensor forms an interesting discussion in itself, but we are limited here to noting that cell 5 is filled.

Again, in passing, we note that the "ordered-pairness" of complex numbers enables us to write a complex function of a single complex variable in a manner identical to that of a vector function of two real variables.

Another useful generalization of the function concept is obtained (cell 3) by considering a function from a set of numbers to a set of statements. Many students are confused to some extent by the nature of expressions of the form $f(x) = g(x)$, where f and g are functions. Take for example the case where $f(x)$ is x^2+x and $g(x)$ is identically zero. We are told that this expression becomes a statement (true or false) whenever the variable x is replaced by one of its possible values. When asked the question, "What is '$x^2+x=0$' before any substitution is made for 'x'?", one might be tempted to reject the question as improperly posed, or to answer merely by giving a name ("equation"?) to the entity. Indeed, this is the most common reaction. If, however, we remember the treatment of function formerly used in high-school texts, we would see that much the same reply was given to the question, "What is a function?" Not equipped with the concept of function in the modern sense, reference would be made to what the function (or expression) x^2 becomes when the variable "x" is replaced by each of a set of possible values. The result of each substitution in the function case was a number. For example, for the function x^2, Figure 3 gives the substitution results for a few cases.

The similarity between the equation case and the function case is so striking that we are tempted to ask the question: "Can an equation in one variable, such as $2x^2-x=x^3$, be thought of as a function of some sort, and if so, what is the domain of this function and the range of this function?" As an aid to visualization, we may compile a table (Fig. 4) similar to the one prepared (Fig. 3) for the function case.

Figure 3

x	x^2
2	4
2.5	6.25
3	9
4	16

x	$2x^2-x=x$
-1	$3=-1$
0	$0=0$
1	$1=1$
2	$2=8$
3	$15=27$
4	$28=64$

Figure 4

We see that an equation in one variable becomes a statement[3] for each value of the variable. For example, when $x=-1$, we get the (false) statement $3=-1$, etc. In this sense, an equation in one variable is a function with a set of numbers as its domain and a set of statements as its range.

The next logical step is to ask whether anything more general than an equation in one variable may be written as such a function. It turns out that any condition[4] on a single variable may be written as a function from a set of numbers to a set of statements. This fact is noted in cell 3. As another special case of condition, any inequality in one variable may be represented as such a function.

Cell 6 bears the same relation to cell 3 as does cell 4 to cell 1 and requires, perhaps, no special discussion. An example of a condition in two variables might be $x=y$. This condition is not to be confused with the *relation* $x=y$,

$$\{(x, y)/x=y\},$$

which is a set of ordered pairs of numbers. (This latter relation also happens to be a function, since to each x there corresponds only one y.) The condition $x=y$ is of the form

$$\{((x, y), A)/\ A \text{ is a statement expressing the equality of } x \text{ and } y\}.$$

[3] A statement has the property of being either true or false. An equation in one variable cannot be said to have either of these properties until a value is inserted for the variable.

[4] A condition may be defined as an expression containing a variable such that when a numerical value is substituted for the variable, a statement results. Thus an equation in one variable is a condition on one variable. The term "propositional function" is frequently used in place of the other.

If we use the word *solution* in its usual sense, we might want to remark that the relation $x=y$ is the solution of the condition $x=y$.

As an example for cell 7, we construct a truth value function, f, which behaves as follows: if A is a true statement, then $f(A)=1$; if A is a false statement, then $f(A)=0$. This function could be an interesting conceptual device in developing the study of logic on the secondary level.

For cell 8, no useful example stands out to the author, although artificial examples may be constructed. Perhaps the reader can suggest a useful function qualifying for cell 8.

For cell 9, we need a function which will, when a statement is "plugged in," give a statement as the result. The negation operator of symbolic logic is such a function:

$\{(A, B)/A$ and B are statements, A is not $-B\}$.

For example, if we let "$\sim$" denote the above function:

$$\sim(3=6)=(3\neq 6)$$

$$\sim(3\neq 6)=(3=6)$$

$$\sim(\text{it is raining})=\text{it is not raining.}$$

In conclusion, we should note that, in setting up Table 1, the choice of numbers, ordered pairs, and statements for discussion is not the only choice we might have made. Columns and rows referring to functions with sets of sets and sets of functions as domain and/or range might have been included. The former is of fundamental importance in theory of probability and mathematical statistics, while the latter finds use in advanced problems in electrical engineering. The reader may be able to think of still other profitable choices for columns and rows.

What's new?

BOOKS

Secondary

Algebra Accelerated, Book I, Justin E. Hills and Estelle Mazziotta. Peoria, Illinois: Charles A. Bennett Co., Inc., 1959. 320 pp., price unstated.

Arithmetic in My World, 9, Newton C. Stokes, Anne Beattie, and Ruth I. Hoffman. Boston: Allyn and Bacon, Inc., 1959. 308 pp. $3.60. Teachers Edition, Teachers Manual.

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College

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Elementary Decision Theory, Herman Chernoff and Lincoln E. Moses. New York: John Wiley and Sons, Inc., 1959. Cloth, xv+364 pp., $7.50.

Elements of Modern Mathematics, Kenneth O. May. Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1959. xvi+607 pp., $6.50.

Theory of Functionals and of Integral and Integra-Differential Equations, Vito Volterra. New York: Dover Publications, Inc., 1959. Paper, 226 pp., $1.75.

Miscellaneous

The Challenge of Science Education, Joseph S. Roucek (ed.). New York: Philosophical Library, 1959. Cloth, 491 pp., $10.

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Meson Physics, Robert E. Marshal. New York: Dover Publications, Inc., 1959. Paper, viii+378 pp., $1.95.

Mathematics program outline for junior and senior high schools

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To help pupils from a wide range of mathematical backgrounds and abilities learn mathematics commensurate with their potential, a wide variety of courses must be offered.

Purpose of the program

THE MATHEMATICS PROGRAM herein described was developed in an attempt to meet the needs of pupils of today—needs emphasized by the obviously wide range of abilities and interests in mathematical areas demonstrated in the classroom. Remedial courses—where students may receive more individual help, may work forward from their levels of achievement, and may concentrate on fundamentals—are provided for those of below-average arithmetic aptitude and achievement; traditional courses are retained for the average; and an accelerated track is set up for those whose abilities lie in the upper strata.

Development of the program

The program is the result of a step-by-step development begun in 1947. At that time ninth-graders were, on the basis of the California Algebra Aptitude Test results, separated into two groups—those who demonstrated an average or better aptitude and were enrolled in first-year algebra, and those who scored less than average and were enrolled in a course in general mathematics, a remedial course.

The success of this venture led to grouping at the seventh- and eighth-grade levels where those who were of low arithmetic ability and achievement were placed in remedial sections. Concurrently, ninth-grade algebra students were enrolled in "homogeneous sections," sections which were determined by testing (aptitude and achievement), eighth-year mathematics grades, and teacher opinions based on such factors as attitudes, interests, work habits, future plans, and regularity of attendance. These classes, ranging from high to low, designated as "Section one, two, three," etc., then progressed within the year of elementary algebra according to the average of the abilities in each section. A year or so later a course called consumer mathematics was added for ninth-graders who might later enroll in the senior-high commercial course, or who might, according to all available evidence, terminate their formal schooling at the ninth grade.

For at least seven years recommendations as to whether a student should elect geometry have been made by the algebra teachers. Again, these recommendations were determined by the algebra grades, intelligence tests, and general scholastic qualities of the student. Three years ago a geometry aptitude test was introduced, the Lee Test of Geometric Aptitude—Form A. It was administered to the students electing geometry, but at this time enough evidence for the practicality of using the test in our school has not been accumulated.

The accelerated program

The apparent success of the arrangements outlined above led the committee to further study and recommendations—this time the needs of the students with

superior mathematics ability were considered.

The first decisions made were to retain the remedial sections in grades seven and eight and to continue to offer elementary algebra, general mathematics, and consumer mathematics in the ninth grade with geometry, higher algebra, solid geometry, trigonometry, and high school arithmetic in the senior high school.

Other considerations included finding a way to offer to the better than average mathematics students more mathematics at the senior-high level, and identifying students of high mathematics ability and enabling these students to include an extra year of mathematics in the high school program.

A decision to start the program in the eighth grade was reached. This decision was based on the following factors:

1. Eighth-grade mathematics teachers had previously been aware that the superior student tended to become bored with the regular eighth-grade mathematics course, for it offered little challenge to him. This situation had often led to an "enrichment plan" that at times seemed merely a "busy work" arrangement to the students.
2. For the sake of continuity and flexibility, it was felt that the students would benefit by having the same teacher for two years, and such an arrangement was entirely practical as an eighth-ninth grade arrangment.
3. Since eventually the junior and senior high schools might not be adjacent to one another, a break between ninth and tenth grades should be provided for in any long term program.
4. In order to offer an additional advanced mathematics course in the twelfth year without forcing the students to sacrifice any other elective subject, it was necessary to compress the traditional five years of mathematics (grades 8 to 12 inclusive) into four years, grades 8 to 11 inclusive.

To accomplish the above, the following plan was put into effect:

1. In an eighth-grade mathematics class of students who had been tentatively selected on the basis of good general ability and high scores on mathematics achievement tests, which had been administered at the end of the seventh grade, the instructor was given permission to move at as fast a rate as the students could handle—testing at the end of each unit to check progress. All parts of the eighth-grade arithmetic course of study were covered except charts and graphs, ratio, proportion, and square root. (These items would be covered in algebra.) At the end of the first quarter the class had covered the essential parts of the eighth-grade course, and the Iowa Basic Skills for Arithmetic (Form 2) was given to the students of this class. Since only thirty-three students had been placed in the class originally, the same test was administered to twelve promising students from other sections—students who had been recommended by their teachers.
2. It is almost inevitable that there will be a certain degree of mortality in this group at the tenth, eleventh, or twelfth grade because of transfers, decisions to enter other areas of study, or possible loss of interest. It was therefore decided to start with thirty-eight students. The median score of the 38 was 10.6 on arithmetic concepts and 11.9 on problem solving.
3. The California Algebra Aptitude Test was given to the same students a day later and the same thirty-eight students, with one exception, scored above the 75th percentile.
4. After the membership of the class was determined, each student was given a letter to take to his parents. The letter stated the purposes, methods of operation, and an outline of the mathematics sequence planned for grades eight

through twelve. The parents were requested to give their approval of their child's enrollment in the course.

5. At the beginning of the second quarter this group started on the regular first-year algebra course. By the end of the year this class had covered as much as is ordinarily covered in the regular ninth-grade algebra classes.
6. Next year, as ninth-graders, members of this class will embark on the study of the regular course in higher algebra under the same instructor.

The sequence of subjects for the accelerated track, as now planned, is shown in the diagram.

A long range look at the program reveals the following advantages:

1. Pupils entering the program stand to gain by having covered, in a given time, more material than would have been possible in a regular group.
2. For them, one year of mathematics will be added to the regular high-school curriculum.
3. There is a possibility, when and if such arrangements can be made, that the mathematics course in the senior year may be taken for college credit.

The parents of the students are very enthusiastic about the program, admit that the students are working harder than ever before, and believe that to be all for the good.

Another group of eighth-graders will be strated on this plan next year. The selection will this time be made at the end of the seventh grade, forty-plus students will be selected, and, if numbers justify it, more than one section will be arranged for.

Flexibility

A degree of flexibility which will permit the student (in cases where it is deemed desirable because of interests or needs) to move from one track to another is maintained in the total program.

At the seventh- and eighth-grade levels there are transfers between the remedial and regular classes, and there can be transfers from the accelerated to the regular classes if necessary.

All students are required to have credit for one of the ninth-grade mathematics courses—general mathematics consumer mathematics, elementary algebra—but when considered advisable, tenth-graders or even eleventh-graders who have not had elementary algebra may enroll in such a class. (Those from the general mathematics classes who have done so, however, have seldom been successful—the prognosis of the screening for ninth-grade classes has been that accurate.)

One course in mathematics in the tenth, eleventh, or twelfth year is required for graduation—the only exception is the substitution of bookkeeping by the commercial students. The required course may be elementary algebra, for those who had general or consumer mathematics at the ninth-grade level, geometry or higher algebra (elementary algebra prerequisite for either), or high school arithmetic. Transfer from the formal or regular mathematics program (the center track in the diagram) to high school arithmetic is possible, and transfer from the general mathematics to the formal program could be arranged in individual cases. Students may enter the accelerated sequence by enrolling in both higher algebra and geometry as tenth-graders, in solid geometry-trigonometry as eleventh-graders, and thereby be prepared to take the advanced course offered in the accelerated sequence at the twelfth-grade level. It is obvious that only the exceptional student could fit into this pattern.

Careful consideration is given to the selection of students for each class. Only the students of superior mathematics ability are enrolled in the accelerated sequence, the better than average and average must take algebra at the ninth-grade level, and those who succeed in elementary algebra are encouraged to follow through, and they usually do, and take geometry or higher algebra or both.

The numbers of sections of each degree of difficulty may vary with the needs of the particular group at any grade level.

Conclusion

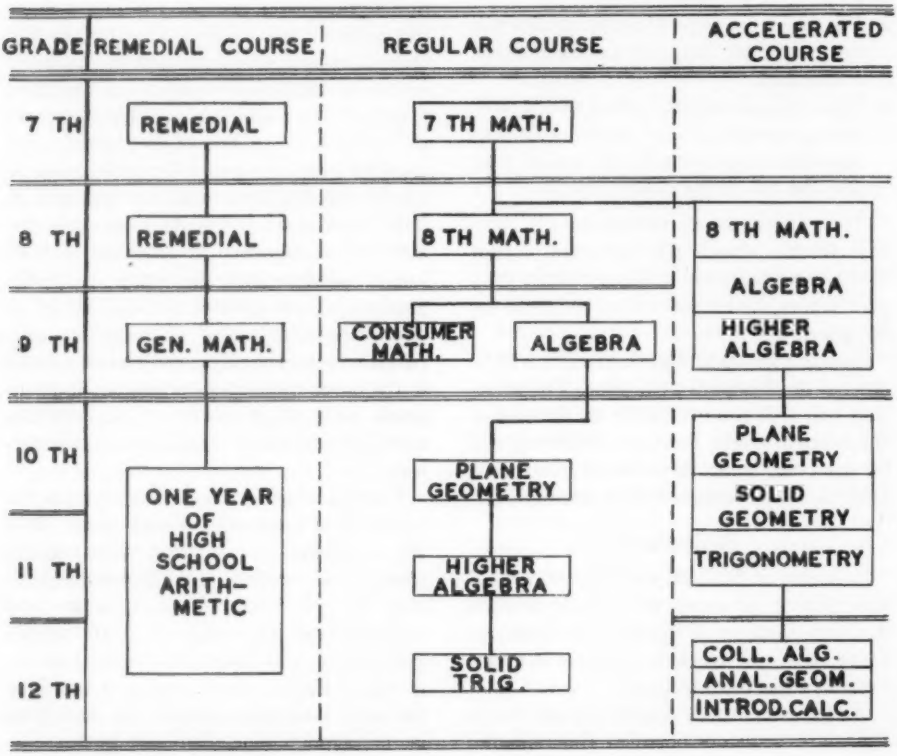
Except for the accelerated sequence, grouping is not emphasized in dealing with the pupils or their parents. There are enough teachers at the various grade levels and in subject matter fields and enough variety in the texts used so that no teacher or text becomes identified with a "slow" or "fast" class. Students are selected at random, therefore are "mixed," for all other subject matter areas (with the exception of certain reading classes) so as not to become identified with each other because of membership in a remedial class.

Teachers, principals, and guidance counselors have combined efforts, and their close co-operation has been effective in creating and bringing about the success of the program.

This program is not a final answer, but it is at present working successfully. It is subject at all times to changes and improvements which may be deemed necessary as a result of further study.

Suggestions, comments, or questions will be welcomed by the Mathematics Department, Junior High School, Stillwater, Minnesota.

PROGRAM OUTLINE
MATHEMATICS
JUNIOR AND SENIOR HIGH SCHOOLS
DISTRICT 834, STILLWATER, MINN.



The evolution of a mathematics contest

RICHARD W. SHOEMAKER, *University of Toledo, Toledo, Ohio.*

Mathematics contests sponsored nationally and locally are gaining in popularity and frequency. Here is a description of how one local mathematics council sponsored a contest.

THE 1956 CONTEST

SOON after the Greater Toledo Council of Teachers of Mathematics came into being in the fall of 1953, a number of projects were suggested for consideration. Among these was an idea advanced by the president, Joseph Jordan of Maumee High School, that the group sponsor a mathematics contest for interested high school students.

The proposal for a mathematics competition was investigated and tentative plans were made by Dr. Harold Tinnappel of Bowling Green State University, Dr. Wayne Dancer, and the author, both from the University of Toledo. The committee was ready to announce plans for a contest early in 1955 when administrative difficulties forced postponement of the project.

By 1956 these difficulties were surmounted, and the local council, under the leadership of President Robert Bolbach, went ahead with its plans. The group knew very little about contests and initiated the event with an exploratory attitude and a desire to learn procedures best adapted to its particular situation. As a starting point there was some literature and correspondence regarding similar experience in the State of Arizona.

Early in February letters were sent to the heads of departments of mathematics of the high schools in the Toledo area. In this announcement the details of the competition were outlined and the number of students in each class likely to participate was requested. Returns from these letters informed the local group that it would have to plan for about 450 contestants. At this time, however, the local press learned of, and publicized, the event with the result that there were 648 registrants. Some students who were not informed by their teachers of the contest were indignant at seemingly being excluded and clamored to be admitted to the competition.

The GTCTM did not charge a registration fee in 1956. The only expense to the students was their transportation to and from the University of Toledo, where the competition was held. The absence of a registration fee was an unfortunate feature administratively, for the sponsoring organization was short of funds. As a consequence some individual members had to make loans for materials and spend long hours in clerical drudgery.

On the morning of the contest, March 24, 1956, Toledo suffered a snowstorm, and only 492 contestants presented themselves. A total of twenty-two schools were represented.

After the papers had been graded a report was sent to each teacher giving the rank and score achieved by his students. In turn the teacher informed the individual student of his marks and explained their significance. The GTCTM has steadfastly refused to supply department chairmen or administrators with this in-

formation, feeling that teachers might hesitate to have their students compete if higher echelon personnel might use the results as a club. The contest is for the students, given by their teachers, and here the matter ends.

Originally the prospect of prizes had been minimized, but as the event approached, prizes materialized from various sources. On April 18, 1956, a short recognition meeting was held to honor the winners and their teachers and parents, and at this time three valuable slide rules, two mathematics dictionaries, and four books of mathematical tables were awarded.

What lessons were learned from this first contest?

Perhaps most significant was the fact that large numbers of young people are interested enough in mathematics to spend a Saturday morning competing with their peers in a mathematical arena in much the same manner that others will compete in football, golf, or debating.

Another benefit was the "glow" felt by members of the GTCTM. Too often mathematics teachers speak vaguely of promoting interest in their subject, but in this contest the Council felt that it had actually found an effective way to accomplish this purpose.

The Council also learned that there are many other activities scheduled for high school students in the spring of the year. Great care must be taken to select a date which will minimize conflicts with athletic events, music festivals, and various other scholarship examinations.

The 1956 contest demonstrated that it was not feasible to hold a contest without a registration fee to pay for stencils, typing, paper, mimeographing, pencils, postage, etc.

The administative work of the project was attempted by too few persons with the result that these persons were overburdened and did not function as effectively as they should have. For later years the GTCTM would employ a much greater number of its members in the project. One committee should have the handling of publicity as its sole duty; another the soliciting or purchasing of prizes; another the processing of registrations; another the arranging of a time for the competition; another the constructing and duplicating of test forms; another the planning of the recognition ceremony for winners, etc.

The 1956 contest was not machine-graded, a circumstance which delayed the return of results to the contestants and which added enormously to the labor involved. Future contests would have to be graded by machine.

With one minor exception there was only one test form for the persons competing at the four scholastic levels in the 1956 contest. The problems were arranged so that the ninth-grade elementary algebra students could solve approximately the first quarter; tenth-grade plane geometry students could work the first half, etc. The exception referred to was that the elementary algebra students, in addition to the regular contest form, took a University of Toledo algebra screening test. This was done to increase the number of problems which the contestants at the lowest level could solve. Winners were selected from each of the four grade levels, and persons in different grade levels did not compete against one another. Comments from various contestants and some teachers indicated that this massive instrument was not ideal for the intended purpose. Despite the extra effort involved, the GTCTM decided that in the future there should be a different examination for each grade level.

The 1957 contest

Encouraged by, and wiser because of, the contest sponsored the previous year, the GTCTM, now led by President Floyd Strow, went ahead with preparations for a second contest. The work was spread over a larger fraction of the organization than the year before; a twenty-five cent regis-

tration fee was charged all contestants; committees were charged with specific tasks in order to faciliate the administration of the project; and four machine-graded tests were constructed. In short, the planning was such as to profit from the experience of the first contest.

The traditional rivalry between Bowling Green State University and the University of Toledo was ignored by the mathematics staffs of these two institutions as they worked together to implement the project. Two of the contest forms and stencils were made in Toledo: the other two and their stencils were made in Bowling Green. All tests were duplicated at Bowling Green, but the competition was held at Toledo. The contest forms were machine-graded by the facilities at Bowling Green.

A total of 1026 students, representing thirty-three high schools, participated in the second contest held March 30, 1957.

On May 1, 1957, recognition ceremonies were held for the winners, and prizes were distributed. Some of these prizes, slide rules and books, were purchased by the GTCTM from part of the registration money. Other prizes, clock radios, thermojugs, glassware, etc., were solicited from local business and industrial establishments. After thinking about the unsuitability of some of these awards, the GTCTM regarded them as the least satisfactory aspects of the 1957 contest and resolved that future prizes would be more in keeping with the nature of the contest.

The 1958 contest

Even before the advent of Sputnik and the resulting furor in science and mathematics education, President Dallas Greenler and the membership of the GTCTM had decided to hold another mathematics contest. With the experience gained in the two previous years the local group felt that it was both wise and possible to extend the scope of the contest by inviting eighth-grade pupils to participate.

Several other changes were made. The entrace fee was raised to thirty-five cents. In order to accommodate the two to three thousand contestants expected, the competition, instead of being held at one place, would be held at four locations: Bowling Green State University, Toledo DeVilbiss High School, Toledo Libbey High School, and the University of Toledo. Having one testing center at Bowling Green would be a great convenience for persons attending from Tiffin, Fostoria, Findlay, etc. The admission tickets were of a different color for each of the five grade levels and matched the color of the test forms for the same level. This color matching was a great timesaver in the matter of uniting students with tests and with all manipulations of the test forms both before and after the competition.

Another innovation for the 1958 contest was the awarding of team prizes. In previous years only individual winners had been recognized, but this year the GTCTM decided that at each level the three persons from a particular school having the highest scores would constitute that school's team at that level. The school's team score was the total of the three individual scores. Team prizes would be given on the basis of the highest team scores.

On April 19, 1958, a total of 2490 contestants presented themselves at the four testing centers. Of this total 1661, representing fifty-six high schools, were from grades nine through twelve; while 829, coming from sixty-four different schools, were at the eighth-grade level.

The 1959 GTCTM Mathematics Contest was held on April 18, 1959. On that day 2914 contestants from ninety-five schools competed at the four locations. Of this total, 824 were from the eighth grade and 2090 from the four upper grades.

Recognition ceremonies for the third contest were held on May 22, 1958, at which time about seven hundred dollars worth of prizes, all but one of which were purchased with registration money, were

distributed to the seventy persons with the highest scores. The individual prizes were slide rules, mathematical tables, and books like *Fun with Mathematics, Romping Through Mathematics, Through the Mathescope, Mathematical Snapshots,* and *Mainstream of Mathematics.* The libraries of schools whose teams placed first and second were given these same books or, if their libraries did not already have it, the four-volume set of *The World of Mathematics.* The mathematics department of one high school was given a demonstration slide rule which had been donated by the Frederick Post Company.

In addition to the prizes, certificates were given to the winners, and honorable mention certificates were sent to 164 other contestants.

Conclusion

The above account describes the still-evolving mathematics contest as sponsored by the GTCTM. The Toledo Council feels that this project is a highly effective way to further interest in mathematics, and that it is by far the most important aspect of the program of the GTCTM. The local council hopes that other groups will be encouraged to undertake similar projects and to do so more efficiently as a result of examining the history of the project in northwest Ohio.

Have you read?

Amir-Moéz, Ali R. "Teaching Trigonometry Through Vectors," *Mathematics Magazine,* September–October 1958, pp. 19–23.

You have been hearing a great deal about trigonometry and the solution of triangles. You may have used other approaches, but the author of this article uses the vector. His demonstration is simple and straightforward. Setting the x axis as the initial side of an angle θ and a vector $\vec{A}$ of unit length as the terminal side permits the x component of $\vec{A}$ to be cos. θ and the y component of $\vec{A}$ to be sin θ. From here the extensions to the other functions and basic identities are simple. Addition of vectors and product of vectors lead to other formulas.

I think you should not pass up this short, but valuable, article. If you do not have time, have your trigonometry students read it.—Philip Peak, *Indiana University, Bloomington, Indiana.*

Coxeter, H. S. M. "Map-Coloring Problems," *Scripta Mathematica,* Memorial Issue 1957, published 1958, pp. 11–25.

The map-coloring problems have fascinated students for years, and there has been much written on the subject. Mr. Coxeter presents in this paper some of the classical proofs but does it in a somewhat simplified manner so that high school students will enjoy reading them. He approaches the problems from the polygons to show how maps of triangles would require only three colors; pentagons, four; and dodecahedrons, four. He shows how Euler's formula of edges, vertices, and faces helps prove the problem. I am sure you and your students will be interested to see how he extends these ideas to the six-color problems of the sphere, the projective plane, or the Mobius strip.

There is a simple development of Heawoods formula as well as color mapping on a torus and Klein Bottle. I think this article will add mathematical insight to your students' concepts.—Philip Peak, *Indiana University, Bloomington, Indiana.*

Hohn, Franz E. "A Mathematical Introduction to Logic Circuits," *Pi Mu Epsilon Journal,* Fall 1958, pp. 400–416.

No one who studies mathematics can escape the study of logic. This article will give the neophyte some insight into the simple steps of logic, such as: what is propositional logic, how do propositions get truth value, what effect do "and," "or," "not" have on truth, what are truth tables, and why do we need "if" and "only if"?

If your students happen also to be students of physics they will be most interested in the connection between electronics and logical functions. Of course, no discussion of this kind would be complete without some thought being given to logic circuits and computations. You will want to read this one.—Philip Peak, *Indiana University, Bloomington, Indiana.*

Chains of reasoning in geometry

JOHN D. WISEMAN, JR., *Tenafly High School, Tenafly, New Jersey.*

The idea of proof as a chain of reasoning may be reinforced by having pupils make chain models of theorems they have proved.

MOST TEXTBOOKS of high school geometry describe the kind of reasoning used in proving a theorem as a "deductive chain of reasoning." Most teachers have undoubtedly used this idea in explaining to their students how steps in proof link together the beginning and the end of the theorem.

However, the student's idea may be that every theorem is a single chain which links together steps in proof as they are written. Also the idea of a chain is not reinforced by the usual method of writing proofs, since statements are separated by lines or are detached from the remaining statements. Thus it seems that something more is needed to really convince the students that they are dealing with an actual chain of reasoning.

There is one pattern of proof which is useful for teaching students about chains of reasoning. This pattern is the hypothetical syllogism.[1] This presents the proof of a theorem as a sequence of if-then statements, as in the following illustration:

If p then q
If q then r
If r then s

If p then s.

This makes the chain evident, for two statements link if and only if the then-part of one is the same as the if-part of the other. Students can see which parts are the same, and thereby gain an idea of how the statements link to form a chain.

This idea of proof as a chain of reasoning may be easily reinforced by having students make chain models of theorems they have proved. An actual chain of reasoning may be constructed in the following manner.

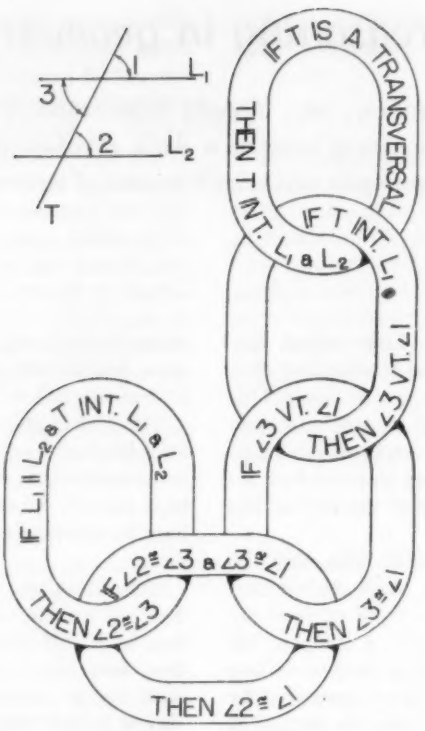
Cut out separate links from paper. Each link, of course, resembles the familiar link in a real chain. Next, take the if-then sentences used in proof and write each on a separate link. The if-part should be written on the top half of the link, and the then-part on the bottom half. Finally, the links are taped together to form the chain.

On page 458 is the chain model for a familiar theorem on parallel lines. For comparison the proof of the theorem also follows, written as pairs of if-then statements. Each specific statement is obtained from the general statement above it by the principle of specification.[2] The general statements form a chain, and the specific statements form another chain. While both of these chains may be combined into one, it is sufficient to make a chain of specific statements only. Of course, prior to the proof of this theorem, it is necessary to prove that alternate interior angles are congruent and that vertical angles are congruent.

While this theorem is not a single chain of reasoning, it is one of the simpler

[1] E. R. Stabler, *An Introduction to Mathematical Thought* (Cambridge, Mass.: Addison-Wesley Pub. Co., 1953), p. 53.

[2] John C. Cooley, *A Primer of Formal Logic* (New York: Macmillan, 1947), p. 94.


1
3
2
L_1
L_2
T
IF T IS A TRANSVERSAL
THEN T INT. L_1 & L_2
IF T INT. L_1 & L_2
IF $\angle 3$ VT. $\angle 1$
THEN $\angle 3 \cong \angle 1$
IF $L_1 \| L_2$ & T INT. L_1 & L_2
IF $\angle 2 \cong \angle 3$ & $\angle 3 \cong \angle 1$
THEN $\angle 2 \cong \angle 3$
THEN $\angle 3 \cong \angle 1$
THEN $\angle 2 \cong \angle 1$

If a transversal intersects two parallel lines, then corresponding angles are congruent.
If $L_1 \| L_2$ & T intersects L_1 & L_2 then $\angle 2 \cong \angle 1$.

If a line is a transversal, then it intersects 2 other lines.
If T is a transversal then T intersects L_1 & L_2.
If the sides of two nonadjacent angles are intersecting lines, then the angles are vertical.

If T intersects L_1 then $\angle 3$ is a vertical angle of $\angle 1$.
If 2 angles are vertical then they are congruent.
If $\angle 3$ is a vertical angle of $\angle 1$ then $\angle 3 \cong \angle 1$.
If a transversal intersects 2 parallel lines, then alternate interior angles are congruent.
If $L_1 \| L_2$ & T intersects L_1 & L_2 then $\angle 2 \cong \angle 3$.
If 2 angles are congruent to the same angle, then they are congruent to each other.
If $\angle 2 \cong \angle 3$ & $\angle 3 \cong \angle 1$ then $\angle 2 \cong \angle 1$.

chains. Many theorems have much more complicated proofs than the one illustrated. In fact, there are many different chains of reasoning used in geometry, and only a few of these are simple, single chains. Making chain models is a way to help students understand the complex patterns involved in these theorems.

It is not intended that the if-then pattern of proof, with the making of chain models, should constitute an entire year's work in geometry. In fact, students need to make only two or three chain models out of paper. Then they should be able to *draw* the chains of reasoning with a considerable saving of time and effort. Other patterns of proof, such as step-and-reason proof, may be used later.

Because abbreviated proofs are used, however, students must have a clear understanding of the meaning of proof and how it is constructed. A chain of reasoning is a means of creating this understanding because it is a model of an abstract idea. Unlike the idea itself, the model is something that the students can see and handle. This means that more students can get a better understanding of proof in geometry.

Reviewing by viewing promotes learning while relearning

ROBERT E. DINKEL, *Culver City Senior High School, Culver City, California.*

An analysis of the educational uses of projects in mathematics teaching.

Introduction

Most teachers agree that students learn readily what they view. An English teacher visited the writer's mathematics room and saw all around the shelves, chalk trays, and window sills a crowded array of student projects—models, manipulative devices, and charts. He remarked, "It's so easy to see how to make projects for mathematics. I wish there were something students in English could do besides notebooks." This greatly surprised the writer since few mathematics rooms show any variety of student projects. The last four years of projects in the writer's room were preceded by nine years with no projects. "What could they make?" Enough, he learned, to clutter up the room with plenty of opportunity for worthwhile viewing!

That visual devices make learning and relearning more palatable is well known; but interest or entertainment is not the real objective. The important fact is that visual devices are of value in promoting *more than scheduled learning*. An intrinsic characteristic of manipulative, visual devices is that they tend to promote *discovery*. The beauty of discovery is that different students can be discovering different things at the same time. This is the hidden implication of our title. If you will use visual devices in teaching, whether for reviewing old material or learning new material, there will always be a bonus. Relationships will be exposed that will fortify former learnings, and will open doors, or windows (or keyholes), to future learnings.

How to become sold on the project approach

To give proper credit, the writer traced back through his background to see where he became sold on the project approach. Early in his teaching experience he discarded the notebook and wondered if extra learning was associated with extra credit given for cutting, pasting, and copying. About 1953 he was doing research on his Master's study, "Algebra Prognosis," at U.C.L.A. This not only re-exposed him to The Mathematics Teacher, but also to the Yearbooks of The National Council of Teachers of Mathematics. This was followed by subscription to The Mathematics Teacher. About the same time curriculum courses helped him discover the chapter by Henry W. Syer on "Sensory Learning Applied to Mathematics" in the *Twenty-first Yearbook*, the section "Laboratory Teaching in Mathematics" in the *Twenty-second Yearbook*, and the article by Emil J. Berger, "Using Concrete Materials as Teaching Aids," in the May, 1954, issue of *The Bulletin* of the National Association of Secondary-School Principals.

Contact with these ideas plus the ex-

amples in THE MATHEMATICS TEACHER in Emil Berger's much-appreciated department, "Devices for a Mathematics Classroom," finally sold the writer on trying it. In the spring of 1955 he passed the word to his students that the following year he would require projects in mathematics. For further preparation, he enrolled in Ida May Bernhard's Mathematics Laboratory at the California Conference for Teachers of Mathematics, held in the summer at the University of California, Los Angeles Campus. This laboratory supplied the encouragement needed; the writer made some 30 models in the eleven mornings of the conference. But he had a surprise in store for him. Few students were content to copy and modify these models; instead they branched out in many directions on their own.

HOW TO SELL THE STUDENTS ON PROJECTS

As in all salesmanship, the salesman must become sold first. A friend remarked that Prof. A in his methods course for mathematics teachers was really sold on the idea of using visual devices. "Did he use them a lot in his teaching?" "Oh no! He wasn't that sold." The backlog of 30 models made in the summer conference made it possible to convince the students that their teacher was really sold. The models were brought out and used as often as possible. When nothing was available, a chalk sketch on the board was used as a starting point for a discussion of how a project could be made that would better show the idea. This analysis as to *how* the idea could be presented tended to bring about visualization which often clarified the idea before the project finally arrived.

RESTRICTIONS ON TYPES OF PROJECTS

For the projects to be successful very few restrictions were made. Luckily, one early requirement was that the students use materials that were financially expendable, since the projects would not be returned. A few materials such as cardboard were available free, but students usually found it more convenient to purchase their needs than to give precise requirements in advance. Many projects were made mainly of scraps found around the house or garage. One reason for emphasizing low cost was a reaction to projects in science fairs running up to $70.00. Probably the maximum cost of materials for any project was about $2.00, though the average, even for the better ones, would be about 50¢ to 75¢.

Disposition of projects was discussed with students in advance. Useful ones would be kept available in the classroom. Duplicates might go to other teachers in the same or different schools. Some of the duplicates were sent to Los Angeles State College for use in teaching of mathematics courses. Dr. Gerald Brown displayed some of the choice ones at Stanford University Summer Session. These and others were shown at the Sixteenth Summer Meeting of The National Council of Teachers of Mathematics at the University of California, Los Angeles Campus. Finally, worn out ones and less useful ones would be discarded.

Since the students saw that their projects would be shared with others, there was no objection to the fact they would not be returned. If any students should object, they could be required to make theirs from school materials. A teacher in another school reported low success with his projects. The crux of the difficulty seemed to be that he promptly returned them, and they were merely stored at home to await a trip to a subsequent teacher for more extra credit.

Subject matter of the projects was almost unrestricted. Although the writer's classes ranged from basic arithmetic to analytic geometry, students at any grade level were allowed the whole realm of mathematics from pre-kindergarten to post-college, as long as the project demonstrated a mathematical concept. This induced *review* or *preview* on the part of the

student when he looked for some basic notion worth demonstrating.

Sample approach of one student

Let us see how one student went about making a project. In passing, it should be noted that the students were advised that it was perfectly legal to have their parents, friends, relatives, etc., help them on (or make) the project. This was done since one is never sure on home projects who did the work, so the honest students were not penalized.

Duane and his dad were trying to hit on an idea. They finally asked themselves for a very basic concept. They hit on the fact that there are 360° of arc on a whole circle. They finally thought of showing this by having 360 small ball bearings arranged in a circle. They decided to make a little trough between two concentric circles the right size to hold these 360 ball bearings. (To figure this circumference and its diameter is a nice measure problem.) To make it easy to count the degrees they decided that alternate groups of ten ball bearings should alternate in color. This was achieved by having half of them plated. This posed an unplanned problem. The increased size of the plated ones now allowed only 353 to fit! They solved the problem by slightly filing the sides of many so that they could fit in the track. The small ball bearings were soldered into the track, and the track was mounted on a board. But the project was not just to look at; brass rods were supplied to represent straight lines, and, by laying these rods across the circle and counting the number of ball bearings in the intercepted arcs, almost any theorem about measurement of angles in a circle could be tested.

Obviously most projects were not this elaborate in construction. Where the student and parent worked together there seemed to be values beyond just doing an assignment. Some students decided it was lazier to do the project alone than to explain to some one else exactly what was wanted, and what it would show. Granted, an occasional one arrived with "My dad made it; I don't know what it shows." This emphasizes the possibility of learning from the completed project. A discussion was held in class, asking what principle was demonstrated. (The class was not told that the submitter did not know.) By the time the class finished listing the hidden ideas, it became a good project.

Requiring and grading the projects

Do not grade the projects! Use them! If you avoid putting an A, B, C on a project you avoid putting a high premium on the expensive or the mechanically-intricate presentation. This helps all the projects to remain respectable. Recognition can be given by leaving them out for students to see. But the best recognition is to *use* the project in class, making mention of the student who presented it. This is worth repeating. Projects must be used, and the student needs the group's appreciation.

Do not require projects from everyone. The first year the writer required one from each student, with a penalty of U in work habits if the project was not in by a certain deadline. This did not work. It meant giving U's where they could not be defended in the total program. The second year a happier plan evolved. Only one project would be required during the entire year. These would be required of only A or B students, with the A student having an earlier deadline.

A very happy bonus resulted. Although about 45 per cent receive A or B in the classes, about 70 per cent turned in projects. The explanation for the large number of projects was not that the students expected extra credit, since it was made clear that the project assignment was a hurdle-type requirement. Doing a project did not make the B student; the B student made the project. Part of the extra number of projects could be explained by the students who did not want to restrict themselves in case they pulled up to a B later. However, in the opinion of the writer, peer status was the underlying

explanation. By handing in a project by the A student deadline, a C student was telling his peers, "For all you know I'm an A student."

The first year one A-caliber student did not make a project and received a U. The second year an A- student got a B at third quarter. He thought it silly that some Algebra II students were doing geometry projects. At the end of the year he brought in a good Algebra II project and got the A. In his project he used different colors on little cards, and he had them arranged in a chart to show the difference between permutations and combinations.

Here is a further comment about evaluation. Each student was asked to make a brief sketch and description of his project on a 5×8 card. Also at each grading period when the student wrote out his self-evaluation he reminded the teacher of his project, again with a very brief description. When a student demanded, the project was orally evaluated with an adjective, never with an A, B, C.

Toy value

Bill, a student in second-year algebra, hit on the quality that really makes the project appeal to other students: toy value. At first the teacher is wary lest all the projects be destroyed, but students cooperate in using the "toys" gently, when reminded. Students like to see things move, and (frustratingly for the teacher) hear them move. A favorite is a model of $a^2+b^2=c^2$ done in small ball bearings or marbles. One has to shake them so that all those from the small squares fit into the large one. This works quite nicely with marbles and a multiple of a 3, 4, 5 right triangle. Also in the same family a model can be made where the area of a circle, πr^2, is shown to equal the area of a rectangle πr long and r high. A quiet model can be made of plastic with the ball bearings replaced with a colored liquid.

After playing with a project, some one is bound to ask, "But what is this supposed to do?" That is when the learning or relearning by viewing takes over. It is refreshing to overhear a student explaining to students from another class just what the projects show.

One project consists of a board with three nails pointing up from the back. There is a set of seven cardboard disks with holes so that they can be stacked, pyramid fashion, on a nail. The object of the game is to move the seven disks to another nail, one disk at a time, never placing a larger disk over a smaller. If n is the number of disks, then 2^{n-1} moves are required of the disks. Gary, or was it Brian, enjoyed this game and started each class period by a timed run. Finally he did it for 7 disks in 1 minute 55 seconds. Then a trap was set for him. An extra disk was made, matched, and fitted into the middle of the pile. This day it took him 4 minutes. He tried again —still almost 4 minutes. When he was told about the extra disk, this was the starting point of a discussion as to how the sum $1+2+4+8+16$ will be doubled (almost) by adding another term—a preview to infinite series in plane geometry class.

Another project with toy value illustrated the ratio of the volume of a cone to that of a cylinder, or of a pyramid to that of a prism, where the solids have equal bases, and/or equal altitudes. This demonstrates how a student can improve on the idea of the teacher. Models are available where a cone can be used three times to fill a cylinder. But this is messy to use. An enclosed system suggested three cones dumping into a single cylinder. A student immediately suggested inverting the cone and using it like a hat on top of the cylinder. Make marks $\frac{1}{3}$ the way up on the cylinder. Fill the system with enough *dry* "fluid" (rice works well, sand is usually too heavy) to fill the cone, and seal. If transparent plastic is used it is easy to invert and see that the volume of the cone equals one-third the volume of the cylinder.

Every project is a good project

A project poorly done may have unsuspected value. A student attempted the cone-cylinder combination model with some cellophane and plastic tape. It happened that the axis of the cone and cylinder did not coincide. Instead of junking it, the model was used to emphasize the fact that the theorem applies whenever the base of the cone agrees with the base of the cylinder, and they have equal altitudes, even when the axis of the cone is quite oblique. This demonstrates another idea in the use of projects. In front of the class, every project must be considered a good project. It is bound to show some mathematics. Privately the student can be shown how to improve the idea.

It is fun to crank Gordon's wire model that revolves a rectangle into a cylinder, or the one that revolves an isosceles triangle into a cone. Any project using elastic thread or hatband elastic is sure to be explored and worn out by the students. A nice model can be made showing the relationship of cylinder, hyperboloid, and cone by threading elastic thread between two concentric disks having equally spaced holes. The disks can be held apart by using a brass rod threaded at each end. Nuts and wing nuts serve to lock the disks in desired positions. Someone is always twisting it.

Of course not all projects develop the appeal of toy value. But, if there is any part of a project that can be moved or adjusted, the interest factor is increased. If it is stationary, bright colors help. For painting, the students have found that rubber base wall paint is very handy since it dries on the way to school the morning the project is due. In general, few students claim spending over three hours actual time working on the project. This does not include the time required in organizing the idea.

Try this idea

A project with most intriguing toy value uses externally a magnet to move internal parts. This has an advantage of being quiet, except for students fighting for the next turn. It is particularly well suited to moving areas as *square* units. The units of area are squares of plywood. Into each is inserted a tack with head on top. The squares are contained in a tray with transparent plastic top. The magnet is used to move the squares. It is interesting to learn that magnets work through plastic or glass. This idea can be used to construct a model of $a^2+b^2=c^2$ that is not open to the objection that can be made when round marbles are used to represent square units.

It can also be used to show the idea of square root. An area greater than 6×36 could be fitted with 36 of these square counters. It would be possible to arrange the 36 square units as 1×36 or 2×18 or 3×12 or 4×9 or 5×7 with one left over or, finally, 6×6. From this final one, the idea of square root based on two equal factors could be explained. A variation is possible for larger numbers. Equally spaced rows and columns of drill points could be drilled in the bottom of a wooden tray. Small ball bearings could be inserted, a plastic cover put in place, and a magnet tied to the end of a string. Again the small ball bearings could be arranged in a corner until they approximate a square. This could give the approximate square roots for an irrational number.

Brief description of other projects

Betty figured out the proper projections of a geometric abstract of squares, triangles, and circles so that when two drawings were viewed through her homemade stereoscope they stood out in 3-D. Joe made a 3-D tic-tac-toe game. He also cut samples of at least 20 geometric figures out of rubber tile, labeled them, and threaded them on a string. He made a cardboard model showing volumes to explain the factoring of $a^3-b^3=(a-b)(a^2+ab+b^2)$. Al made a cubic yard out of welding rod. It was so big it had to be cut

to get through the door. Yarn was used to connect the insides to show the 27 cubic feet. Few of us realized how big it would be. Later we got a wooden model of a cubic foot supporting a cubic inch. Bob made us a model of a transit to help the class watch plane crashes. (The class actually had seen the Pacoima airplane crash from about 20 miles away, but could not tell the angle of elevation.) Dick made an Egyptian level which works with a plumb bob.

Rex made a tremendous project that showed vector addition. He had a plastic sheet that slid diagonally on the background. On the sheet was a tiny airplane with a track on the plastic. Since both could be moved, vector addition for these fixed angles was possible. Sandy found out about the pseudosphere and made one out of papier-mâché from a belled vase as a form. An old beach ball served as a sphere. On each he drew triangles that showed the sum to be less than 180° on the former, and over 180° on the latter. (Here the toy value interested the class more than the non-Euclidean geometry.)

A handy hemisphere is the silvered end of a large light bulb. Pencils write on it well. An easy ellipse can be made by cutting a cylinder on the diagonal. Building blocks can show what happens to the ratio of volume to area as a solid is enlarged, giving the idea of critical mass. But you will never use these items if you do not get them into the classroom *before* you need them there.

Finale

There are several selfish bonuses for the mathematics teacher with a room cluttered with projects. The kids seem to like to come into the room. Parents and administrators are pleased at open house. But the crowning one is the fact that the teacher no longer can be accused of needing only a textbook, an answer book, and the corner of a different room for each period of the day.

> " . . . Teaching aids must be used at the right time and in the right way if they are to be effective in promoting learning. Failure to select appropriate teaching aids and failure to use them properly can destroy their effectiveness. The excessive use of teaching aids may lead to overdependence on physical representation and inability to work with the symbolic representation of an abstraction. The improper use of teaching aids may create confusion and misconceptions. The indiscriminate use of teaching aids may occasionally result in pure entertainment. The question one must ask himself when selecting and using a teaching aid is: what unique contribution will this material make toward better learning that cannot be attained without it? Teaching aids should not be used merely because they are desirable and popular, or because the material is available, or because the class period has to be filled in. We must recognize that materials are only part of the learning situation and are always subordinate to aims and methods."—*Emil J. Berger and Donovan A. Johnson,* "A guide to the use and procurement of teaching aids for mathematics," Bulletin, National Council of Teachers of Mathematics, April, 1959, p. 4.

Secondary school mathematics in the Federal Republic of Germany

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A description of the mathematics program required of all secondary school students in Germany.

The educational system of the Federal Republic of Germany[2]

Education in the Federal Republic of Germany is the responsibility of the individual States. There is no Federal Ministry of Education. With the exception of the Nazi period, 1933–1945, when a National Ministry of Science, Education, and Public Instruction was established, this has been the historical pattern of German education.

Every graduate of a general secondary school is automatically eligible for admission to any university in the Republic. Uniformity of requirements for the secondary school leaving certificate is therefore a major objective of all the States. To achieve this uniformity as well as to provide for general co-ordination in educational and cultural matters, the ten States and West Berlin have formed an association known as the "Permanent Conference of the Ministers of Culture," with a full-time secretariat in Bonn.

The ministers themselves or their representatives constitute the membership of the Conference. To assist the Conference in its work, three general committees have been established: the School Committee, the University Committee, and the Cultural Committee. Each State designates its own representatives to sit on these committees. The Conference meets monthly for two or three days; the committees are in session for whatever period is necessary to cope with the problems placed before them.

The typical secondary school in the Federal Republic is a nine-year institution, including grades 5–13. Upon completion of the fourth grade of the elementary school, approximately 75 per cent of the pupils continue in the elementary school for an additional four or five years, about 7–8 per cent enter the middle schools (usually grades 5–10) and the remaining 17–18 per cent enter the secondary school. The upper elementary, the middle, and the secondary schools are three distinct parallel types, each with its own curriculum, and leading to distinct vocational or professional objectives. The objective of the general secondary school is to prepare its students for admission to the university. There are a few special secondary schools, such as the Commercial Secondary School (grades 10–13), which have a direct vocational objective.

The Permanent Conference of Ministers of Culture has adopted the name "Gym-

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[2] A detailed study of the German educational system was published by the Office in 1957. Alina M. Lindegren, *Germany Revisited, Education in the Federal Republic*, Office of Education, Department of Health, Education, and Welfare, Bulletin 1957, No. 12. (Washington: U. S. Government Printing Office.)

nasium" for all general secondary schools. Three distinct types are common: the classical language, the modern language, and the mathematics-science types. The greater part of the curriculum is identical in all three. Variations occur only in the subjects indicated in the titles. There are practically no electives. When a student enters any given school, he has selected his curriculum for the entire nine years. Only about 25 per cent of the students who enter the secondary school in the fifth grade complete the entire nine-year program.

Several variations from the standard nine-year secondary school should be noted. For many years, a considerable number of educational theorists in Germany have advocated a longer period of common elementary education for all children. Bremen, Hamburg, and Berlin have moved in this direction. In Bremen, all pupils remain in the elementary school through the sixth grade. The secondary school therefore includes grades 7–13. Hamburg had adopted the same pattern in 1949 but modified it in 1955 so that pupils can now enter the secondary school either after the fourth or after the sixth grade. Berlin adopted an eight-year common elementary school in 1948, but reduced it to six years in 1951.

Another variation that is found in most States is the "Gymnasium in short form," developed during the past quarter century to provide an opportunity for secondary education to children who developed late and therefore had not been admitted to the secondary school at the end of grade 4. It usually is a six or seven-year institution based on seven or six years of elementary school respectively. The ultimate objectives of all general secondary schools, terminating with the thirteenth grade, however, are the same. The leaving certificates of all of them are equal in value. A general resolution of the Permanent Conference dealing with mathematics education[3] recommends that schools beginning later than the fifth grade should plan their subject matter in such a way that the pupils will reach the accepted standards of the nine-year school by the end of the tenth grade, thereby assuring equivalency of instruction in all types of schools at the upper level (grades 11–13).

The mathematics program in Baden-Wuerttemberg[4]

Many decades of tradition, the lack of electives in secondary school programs, the similarity in the training of all secondary school teachers, as well as the coordinating work of the Permanent Conference have led to a high degree of uniformity in the mathematics program throughout the Federal Republic. A discussion of the program in Baden-Wuerttemberg will therefore serve as a pattern for the entire country. Variations from this pattern are slight. A student from any one of the ten States could transfer without difficulty at his appropriate grade level into the program as described here, or could participate in the final examination in any one of the ten States or in West Berlin.

Mathematics in the total curriculum

The total school program varies from 28–29 hours per week in grade five to 32–33 hours in the upper grades. Table I indicates the extent of mathematics instruction.

In the classical language type, the hours devoted to mathematics are exceeded only by German (35) and Latin (47); in the modern language type by German (35),

[3] Beschluss der Kultusministerkonferenz vom 25.3.58 betr. Richtlinien und Rahmenplaene fuer den Mathematikunterricht. Bonn. Staendige Konferenz der Kultusminister. Secretariat. Nr. 2350/58. (Resolution of the Conference of Ministers of Culture of March 25, 1958, regarding Guidelines and skeleton plans for mathematics instruction.)

[4] The pre-war States of Baden and Wuerttemberg were divided into three occupation areas in 1945, as follows: Wuerttemberg-Baden in the U. S. Zone, and South Wuerttemberg-Hohenzollern and South Baden in the French Zone. In 1953 these subdivisions formed a single State under a single constitution and adopted Baden-Wuerttemberg as the official name.

TABLE I

HOURS PER WEEK DEVOTED TO MATHEMATICS IN THE SECONDARY SCHOOL OF BADEN-WUERTTEMBERG[5]

Type of Secondary School	*Grade*									*Total*
	5	6	7	8	9	10	11	12	13	
Classical language	5	5	4	4	3	3	3	3	3	33
Modern language	5	5	4	4	3	3	3	3	3	33
Math.-Science I[6]	5	5	4	4	4	4	4	4	5	39
Math.-Science II[6]	5	5	4	4	4	4	4	4	4	38

Latin (45), or English (38), depending on which language the student selects in Grade 5; in the mathematics-science type, mathematics is approximately equaled by German (38) and English (38). Essentially the same mathematics subject-matter is covered in all three types of schools and the same textbooks generally are used. Instruction in the mathematics-science type, however, is more intensive. The individual topics are pursued at greater length, more difficult problems are used, and a more difficult final examination is given. The procedure could be compared to the use of starred paragraphs and problems in some American textbooks, making it possible to select courses of varying difficulty from the same text. In the Federal Republic, however, each teacher makes his selections on the basis of the detailed guidance provided by the State course of study.

THE COURSE OF STUDY[7]

The State course of study includes a section on purposes and objectives, one on teaching methods, and a general statement on each of the three major areas: arithmetic, algebra and analysis, and geometry. These are followed by a list of topics for each grade. The primary goal is to develop an understanding of mathematics as a purely intellectual system as well as to become familiar with its applications to the real world. Some of the specific goals are: skill in working with figures and forms; mastery of the rules, theorems and laws of school mathematics; understanding of spatial relationships; training in logical thinking and in the development of exact concepts; the development of brief and exact expression; the development of habits of thoroughness, persistence, order and objectivity; experiencing the pleasure of independent work and discovery; sensing the inherent beauty of mathematics; the ability to recognize the mathematical aspects in the manifestations of the surrounding world; an insight into the foundations and logical structure of mathematics; an appreciation of the philosophical and cultural content and importance of mathematics.

Comments on method emphasize the inductive approach to new materials. Pupils are to be encouraged to raise questions, to suggest solutions, and to attempt independent proofs. The concept of function is to be stressed throughout the program. Concepts of sets, fields, and groups are to be introduced with simple examples. Where scientific rigidity of proofs must be omitted because of the level of the students, the students should be aware of these omissions. Memorization should be kept to a minimum. Relationships with other subjects in the curriculum must be stressed.

The subject matter for each grade is outlined as follows (science-mathematics types):

Grade 5: Review and extension of computation with whole numbers; money; length, area, volume, weight, time; decimal notation and application in addition and subtraction; simple fractions from everyday life; graphical presentation of quantities; introduction of the

[5] Kultusministerium des Landes Baden-Wuerttemberg; Lehrplaene fuer die Gymnasien Baden-Wuerttembergs, 4 Februar 1957. Villingen: Neckarverlag GMBH. (Ministry of Culture of the State of Baden-Wuerttemberg; Course of Study for the Secondary Schools of Baden-Wuerttemberg, February 4, 1957.)

[6] Types I and II differ very slightly. The additional hour of mathematics in Type I is offset by a decrease of one hour in foreign language.

[7] See reference No. 5.

parallelepiped, rectangle, cube, and square; introductory concepts of geometry; use of drawing instruments; estimating and measuring; building of models; divisibility of numbers; prime factors with use of exponents to designate repeated factors.

Grade 6: Computation with common and decimal fractions; conversion of common fractions to decimals and vice-versa; discussion of the cylinder, circle, angle, prism, regular polygon, pyramid and equilateral triangle, cone and sphere; computations with direct and inverse proportion.

Grade 7: Review of fractions; percentage with simple applications to the practical problems of individuals and of society; projection on an axis, translation and rotation; elementary geometry problems; introduction to the study of the triangle and the quadrangle; introduction to algebraic numbers; simple linear equations with one unknown.

Grade 8: The four fundamental operations with rational numbers; linear equations with one unknown; continuation of the study of the triangle and the quadrangle; congruence; area of figures bounded by straight lines; volume and surface of simple prisms; lines and angles on a circle; the function concept; the linear function.

Grade 9: Linear equations with two unknowns, algebraic and graphic solution; theorem of Pythagoras; square root and irrational numbers; use of tables; the functions $y = X^2$ and $y = \sqrt{X}$; equations of the second degree with one unknown; the general quadratic function; symmetry; similarity; powers with integral positive and negative exponents; the function $y = ax^n$.

Grade 10: Powers with fractional exponents; exponential and logarithmic functions; logarithms; use of logarithmic tables and the slide rule; computations of the circle; the number π; volume and surface area of the cylinder, pyramid, cone, and sphere; construction of figures in vertical and skewed projection; trigonometric functions of the acute angle; computations with right triangles.

Grade 11: Trigonometric functions of all angles; sine and cosine equations with applications; mensuration; the function $y = a \sin (bx+c)$; the concept of vectors; multiplication of a vector by a scalar; sum, difference and scalar product of vectors; introduction to analytic geometry; the straight line; arithmetic and geometric progressions with simple applications; infinite geometric series; the concept of limits and computation with limits; the number e; introduction to differential calculus with integral rational functions.

Grades 12–13: Continuation of differential calculus; fractional rational functions, power functions, trigonometric, exponential, and logarithmic functions; simple geometric and physical applications; introduction to integral calculus; areas of plane figures and volumes formed by revolved functions; examples of the application of the integral to physics; analytic geometry of the circle; discussion of conic sections by a variety of methods, emphasizing projections; geometry of the sphere with applications to the earth and to astronomy, using graphical presentation whenever possible; extension of the concept of the number system to include natural and complex numbers; cultural and philosophical aspects of mathematics.

Textbooks

A large number of new textbooks have appeared in recent years. All textbooks used by the schools must be approved by the Ministry of Education. In Baden-Wuerttemberg, the Ministry does not prescribe a specific text, but has approved four series from which the local school systems can make their selection. Most textbooks are written in a series including grades 5–13. The similarity of the mathematics program in the various States is further evidenced by the fact that many textbook series have several joint authors representing a number of States.

Most textbooks differ considerably from American texts. They are essentially collections of problems with a minimum of explanatory material. It is the function of the teacher to supply whatever additional material is needed to work the

problems. In the few classes that the writer was able to observe, he noted considerable use of the lecture method to present new subject matter. The inductive approach to particular problems, however, appeared to be quite common. In general, the textbooks are much briefer than an American text dealing with the same amount of subject matter.

The following table presents an analysis of the textbooks used most frequently in Baden-Wuerttemberg:

TABLE 2

THE LAMBACHER-SCHWEIZER MATHEMATICS TEXTS[8]

Title	*No. of Pages*	*No. of Problems*	*Grade*
Arithmetic & Space Study I	162	1270	5
Arithmetic & Space Study II	184	1448	6–7
Algebra I	125	758	7–8
Algebra II	119	619	9–10
Geometry (E) I[9]	91	585	7–8
Geometry (E) II	144	811	9–10
Plane Trigonometry	80	386	10–11
Analytic Geometry	180	967	11–13
Analysis	216	1087	11–13
Spherical Geometry	64	202	12–13
Total	1365	8133	

Table 2 illustrates the German, and, in fact, the typical European approach to a difficult problem in mathematics instruction: the continuity of experience. Grades 5–6 emphasize arithmetic and the beginning of intuitive geometry through the study of geometric figures, areas, and volumes. In grades 7–10, arithmetic is carried forward into algebra, and a systemmatic study of algebra and demonstrative geometry parallel each other. Plane trigonometry overlaps algebra and geometry in grade 10 as well as analytic geometry in grade 11. Analytic geometry and calculus parallel each other in grades 11–13. Finally, spherical geometry is introduced in addition to the above in grade 12 or 13.

[8] Lambacher, Theophil and Schweizer, Wilhelm, *Mathematisches Unterrichtswerk fuer Hoehere Schulen* (*Mathematics Text for the Secondary Schools*) (Stuttgart: Ernst Klett Verlag, 1957).

[9] Two series of geometry are available. The E series is Euclidean geometry with the usual demonstrative approach. The A series, used frequently in high schools for girls, covers essentially the same material but is more intuitive and places less emphasis on formal proof.

EXAMINATION PROCEDURES

No final examinations are given at the end of each year. However, during the year there are a number of supervised periods in which the student, without any assistance from the instructor, must work a group of assigned problems in a given time. Ten such periods are scheduled in each of grades 5 and 6, 8 in grades 7–12, and 6 in grade 13. The instructor assigns a yearly grade on the basis of these supervised periods, the student's homework, and his general participation in class.

The secondary school closes with a comprehensive examination covering all the basic subjects taken in the nine-year period. The mathematics examination in all three types of secondary schools usually consists of a choice of three out of four problems with a time limit of four hours. The level of difficulty for the mathematics-science type will be somewhat higher than for the others.

The following problem, taken from the examination given in Baden-Wuerttemberg in 1957 illustrates the typical level:

Given is the function $y = ex + e^{-x}$

a. Determine the minimum point and the asymptote of the curve. Show that there is no point of inflection.
b. Draw the curve between $x = -1.5$ and $x = +1.5$ by means of the addition of ordinates.
c. Compute the area enclosed by the curve, the x-axis, and the y-axis.
d. Compute the area enclosed by the curve, the y-axis, and the asymptote.

SUMMARY

Secondary school mathematics in the Federal Republic of Germany, as illus-

trated by the above description of the curriculum in the nine-year, mathematics-science secondary school of Baden-Wuerttemberg, is an intensive, continuous program from the fifth through the thirteenth year. It is one of the two subjects, German being the other, in which all students are examined upon completion of the secondary school. Continuity of experience is maintained, not only by requiring mathematics in every school year, but by parallel instruction in several subjects for relatively long periods of time.

Upon completion of the program, the student has a thorough introduction to elementary mathematics including differential and integral calculus. The level of achievement appears to compare favorably with that of an American student who has taken mathematics every year in high school and has completed a thorough course in analytic geometry and calculus in college.

A number of leading German educators whom the writer interviewed believe that the mathematics requirements have been overemphasized. A few believe that some students who are gifted in other fields are driven from the secondary school by these requirements. One leading official in a State Ministry of Culture stated that the final mathematics examination for the school-leaving certificate is the most feared hurdle for the majority of the students. In his opinion, analytic geometry and calculus should be postponed to the university. There is no evidence, however, that any considerable number of educators support this view. The current program, therefore, will undoubtedly remain in effect for some time to come, with only minor, if any, revisions.

From "The Mathematics Teacher" of thirty years ago

"Those of our educators who continually work against mathematics should keep in mind that with mathematics, as with any branch of learning, its practical and cultural value must be measured not by whether one can get along without it if one does not know it, but by whether and how much of it one will use if one does know it."—*John A. Swenson*, "The Newer Type of Mathematics."

". . . The traditional computational introduction to informational mathematics is clumsy and ineffective. The hard crust of tradition which for generations has covered and obscured the real essence of mathematics must be removed. When one recognizes the historical fact that a long procession of textbooks in mathematics has followed antiquated models so faithfully that it is almost impossible to tell them apart, one realizes that a revolution may be necessary to disturb the complacent slumbers of the teachers and textbook makers in this field. Perhaps American schools will have to be deprived for a time of the full benefit of the kind of mathematics now current in order that mathematicians may be persuaded to mend their ways."—*Charles H. Judd*, "Informational Mathematics Versus Computational Mathematics."

A punch-card "adding" machine your pupils can build

LAREW M. COLLISTER, *Maine Township High School West, Des Plaines, Illinois.*

An easily constructed device to use in teaching binary number notation and intersection of sets.

BEGIN WITH A STUNT. Take six large poster-size cards, numbered as in the table below. For convenience refer to them as "*A*," "*B*," etc.

Ask someone to think of a number between one and sixty-three, without telling what number he has chosen. Then ask him successively, "Is it in *A*?" "Is it in *B*?" etc., and finally surprise him by telling him what number he had selected. (The secret, of course, lies in adding the number which appears in the upper left corner of the cards to which he answered "yes," the sum being the number selected.)

Of course you can have lots of fun with this before you tell (or your pupils discover) *how* to work the trick, but more pleasure follows in the discovery of the *why*. The punch-card "adding" machine will help make it clear. Take sixty-three cards, 4″×6″, and punch six holes along one edge of each card. These holes should be uniformly spaced laterally and at a distance from the edge equal to the diameter of the hole. The cards must be punched in such a way that, when evenly stacked and held in position, a wire or ice pick or some such tool will go easily through the deck. It is useful, too, to cut a triangle from one of the corners for purposes of reorienting the cards after any shuffling that occurs.

Number the cards from one to sixty-three in large characters that will be easily read from the back of the room. It

A				*B*				*C*			
1	17	33	49	2	18	34	50	4	20	36	52
3	19	35	51	3	19	35	51	5	21	37	53
5	21	37	53	6	22	38	54	6	22	38	54
7	23	39	55	7	23	39	55	7	23	39	55
9	25	41	57	10	26	42	58	12	28	44	60
11	27	43	59	11	27	43	59	13	29	45	61
13	29	45	61	14	30	46	62	14	30	46	62
15	31	47	63	15	31	47	63	15	31	47	63
D				*E*				*F*			
8	24	40	56	16	24	48	56	32	40	48	56
9	25	41	57	17	25	49	57	33	41	49	57
10	26	42	58	18	26	50	58	34	42	50	58
11	27	43	59	19	27	51	59	35	43	51	59
12	28	44	60	20	28	52	60	36	44	52	60
13	29	45	61	21	29	53	61	37	45	53	61
14	30	46	62	22	30	54	62	38	46	54	62
15	31	47	63	23	31	55	63	39	47	55	63

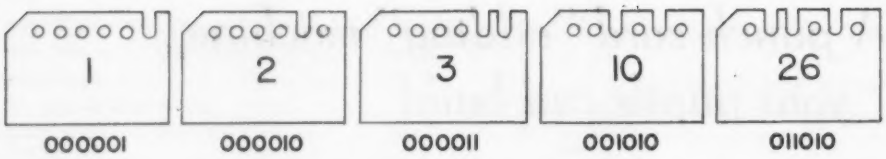


Figure 1

is handiest if the numbers are printed with the punched edges up.

The problem now is to encode the numbers written on the cards, and for this purpose we use the binary number system, letting each of the six hole positions be a place holder for the digits 0 or 1. We cut a V-shaped notch to represent 1 at a given position, or leave the punched hole simply as it is to represent 0. Some examples are shown in Figure 1.

When a wire is run through the deck and the deck shaken lightly, the cards notched for that position will drop away, while the others will remain on the wire. Call the hole position farthest to the right "*A*," next "*B*," etc., as shown in Figure 2. Put the wire through hole *A* and carefully shake out all the cards notched for that position. The set of cards that drops away is exactly that listed on poster *A* above. This suggests a method of sorting out a pre-selected number by a series of sorts, each time taking the set of cards for the next sort from the notched (fall-outs) cards if the answer is "yes," or from the unnotched (on the wire) cards if the answer is "no" to the corresponding question, "Is it in *A*?" etc.

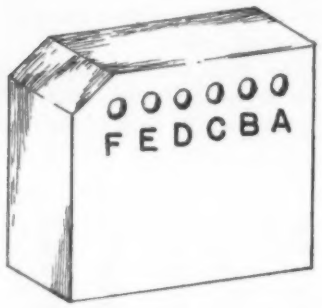


Figure 2

Take an example. Suppose the number is 26. You are accordingly told "no" to your question about the *A* poster. Put the wire through the hole farthest to the right, shaking loose all the cards which are notched for that position. The answer was "no," so you discard the notched group. The group remaining is sorted again. The number 26 *is* in the *B* set, so when you insert the wire through the *B* hole and shake out the loose cards, *keep* the ones which are notched (drop away). The number 26 is not in *C*, so in the next sort, keep the cards which do not drop away. Since 26 is in *D* and *E*, keep the notched cards. You have left just two cards, and since 26 is not in *F*, keep the single card remaining on the wire from the last sort. You will not be surprised, but your pupils will, that this number turns out to be 26.

It may be somewhat more enlightening to tabulate the first twenty-five or so integers in decimal notation next to their binary representations. In any vertical column of digits in the binary system, notice the occurrence of a 1, and the corresponding occurrence of that number's decimal equivalent in the original *A*, *B*, *C* posters.

In the preceding example you have, in fact, "added" $2+8+16=26$, and any generous person would be willing to say you had an "adding" machine. Of course the possibilities of this adding machine are somewhat limited, but it will find various intersections, unions, and complements of sets in this universe of 63 elements. For indeed, if the universe consists of the first sixty-three natural numbers, and the sets *A*, *B*, *C*, *D*, *E*, and *F*

are tabulated in Figure 1, then

$$\{26\} = \overline{A} \cap B \cap \overline{C} \cap D \cap E \cap \overline{F}.$$

The set $A \cup B$ is the set of cards which drop away on the A sort plus the cards which drop away on the B sort. $A \cap B$ is the subset of $A \cup B$ which falls away first on an A sort, then on a B sort. Other examples useful in discussions of sets will occur to you in connection with this exercise.

Another stunt you might try is to sort the punched cards from any random arrangement into numerical order. This is done by starting the sort again at A, but putting the notched cards at the back of the stack, continuing to B, C, D, etc. When the sixth sort is completed, the cards will be in the required order. An article entitled "Card Sorting and the Binary System" by John E. Milholland in the section "Mathematical Miscellanea," edited by Phillip S. Jones, appearing in THE MATHEMATICS TEACHER, May, 1951, pp. 312–314, discusses card sorting further, and mentions some of the commercial applications of punch-card techniques, as well as some references.

From "The Mathematics Teacher" of thirty years ago

"The most important problem in our present system of mass education is that of adjusting instruction to suit the individual pupil. Our uniform 'whole-class' method of instruction is designed for the 'average pupil.' The tendency to group together pupils of widely differing abilities brings about a situation where a lesson, which is too short for some, is too long for others. The more gifted pupil is deprived of the opportunity for his full development since he is not called upon to do his utmost, and therefore marks time. The less gifted pupil, on the other hand, being forced to compete with the brighter, becomes discouraged, develops the feeling of inferiority and habits of failure."—*H. Weissman*, "Grouping in Geometry Classes."

"I am sure we all agree that the appreciation of the beautiful, whether in art, in nature, or in science, contains elements of vast cultural value. It is my impression that this aspect has not been sufficiently emphasized in connection with mathematics. It is taken for granted, of course, that mathematics will appear beautiful to him who devotes his life to this exacting mistress, but it also seems to be assumed that the beauties of this science are by their nature hidden from the eyes of the layman. . . ."—*Aubrey J. Kempner*, "The Cultural Value of Mathematics."

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

Simon Stevin and the decimal fractions

by D. J. Struik, Massachusetts Institute of Technology, Cambridge, Mass.

1

Simon Stevin, the Flemish-Dutch engineer and mathematician, is best known in the history of mathematics as the inventor of the decimal fractions. This is substantially true, even if we acknowledge the fact that decimal fractions were used before Stevin. It was primarily through Stevin's little book, really no more than a pamphlet, entitled *De Thiende* (*The Tenth*), published at Leiden in 1585, that decimal fractions became a regular part of the curriculum in arithmetics.

This pamphlet was republished, in Dutch, French, English, and Latin, several times during Stevin's lifetime and shortly thereafter. Its text has now been made available again in a photostatic reproduction, together with a contemporary English translation by Richard Norton, as part of the second volume of the new edition of Stevin's principal works, prepared under the auspices of the Physics Section of the Royal Netherlands Academy of Sciences [1].* This edition contains an introduction devoted to the history of decimal fractions which gives us an opportunity to evaluate Stevin's achievement in this area.

Simon Stevin was a native of Bruges; it is fairly certain that 1548 was the year of his birth. He started as a bookkeeper, but left Flanders in the period of unrest and persecution which opened the war with Spain. In 1581 we find him at Leiden, and from that time until his death he lived in Holland. Here, like thousands of other immigrants from the Southern Netherlands, he had to find a living under new circumstances. With his particular talents as engineer, accountant, and mathematician, his welcome in the new and striving republic was assured. For many years he was an adviser to Prince Maurice of Orange, one of the most accomplished military commanders of his age, whom he served not only as military engineer, but also as a tutor in different fields of mathematics. Much of Stevin's later work bears the imprint of these tutoring sessions, which covered a wide field of practical and applied geometry, trigonometry, perspective, and bookkeeping. After his death at La Haye in 1620, his admirer Albert Girard, also an immigrant mathematician and engineer, edited his collected mathematical works in French [2]. This edition was published as a big folio in 1634, and has been the main source of our knowledge of Stevin as a mathematician until 1958 [3]. The new edition of *The Principal Works* is planned in five volumes, of which Volume I, dealing with Mechanics, and Volume II, dealing with Mathematics, have already appeared. When the five volumes are published they will offer an excellent insight into the status of the exact sciences in Europe in the period just before Fermat and Descartes opened the new era of calculus and coordinate geometry.

A decimal fraction has a positive integral power of 10 in its denominator, and

* Numbers in brackets refer to the notes at the end of the article.

can be written in many ways, e.g.: 71/100, 0.71, 0,71, or .71, to use notations in use at the present time. It is not easy to say where decimal fractions were first used in any systematic way, but the priority seems to go to the Chinese [4]. With them we find the decimal place-value notation as early as the 14th century B.C. and the use of decimal fractions in metrology as early as the 3rd century A.D. Liu Hui, who lived in this period, expressed a length of 1.355 feet as 1 chhih, 3 tshun, 5 fen, 5 li. The same Liu Hui also performed root extractions in decimals; in this process, which we find in the Middle Ages in use among Arabic, Jewish, and Latin authors, one writes, for instance, $\sqrt{17} = \sqrt{170{,}000}/100 = 412/100$. When we come to the Sung Dynasty, we find computation with decimal fractions already well developed; Yang Hui, in 1261, multiplies 24.68 by 36.56 and finds 902.3008. From the Chinese and the Indians the decimal notation came to the Islamic writers; our example of $\sqrt{17}$ is found in the writings of Al-Nasawī (Persian, ca. 1030), who translated 412/100 again into sexagesimal fractions as $4°7'12''$, meaning $4+7/60+12/3600$. The familiarity of Yang Hui with decimal fractions was shared, one and a half centuries later, by the Persian astronomer Jamshid Al-Kashi [5], who multiplied 25.07 by 14.3 to obtain 358.501. Whether the first notion of decimal fractions in Europe came through contact with the Orient or whether it arose spontaneously is difficult to say, but there is little doubt that wherever computations in the decimal system were used on a wide scale, decimal fractions were sooner or later bound to appear through the logic of the computations themselves. As it was, it is rather astonishing that with the decimal system in use since the Stone Age in so many parts of the world, the regular use of this system for fractions appeared at so late a period.

Readers may ask: if decimal fractions appeared outside of China at such a relatively late date, how did people get along before their introduction? The answer is that they used either fractions like 3/7, 7/13, etc., with any kind of denominator, or the sexagesimal fractions based on the number 60. An example of a quite complicated reckoning of the first kind is Archimedes' approximation of π as a ratio between 3 10/71 and 3 1/7; examples of sexagesimal reckoning can be found on cuneiform clay tablets dating back to Iraq (Mesopotamia) of the fourth millenium B.C. Ptolemy, in his astronomical handbook known as the *Almagest* (ca. 150 A.D.), also used sexagesimal fractions. And so do we when we express the magnitude of an angle as $29°51'32''$; here minutes and seconds are expressed in the sexagesimal system, while the numbers 29, 51, 32 are written in the decimal system. Variations of these two methods of the fractional calculus are known to have existed, such as the ancient Egyptian method of expressing fractions as the sum of unit fractions (fractions such as $\frac{1}{3}$, $\frac{1}{7}$, with numerator 1). Strictly speaking, there was, and still is, another way of coping with the problem, popular wherever fractional calculus is considered difficult, and that is the avoidance of all fractions by the choice of an appropriate scale of measurement, e.g., 60, so that $\frac{1}{4}$ is expressed by 15, $\frac{3}{8}$ by 24, etc. We do the same when we say 475 m instead of .475 km, or 3 quarts instead of $\frac{3}{4}$ gallon. We shall see how all these concepts played a role in Stevin's work.

2

During the fifteenth and sixteenth centuries, European computers began to use the Hindu-Arabic system of decimal notation—that is, our present positional system—with ever-increasing efficiency, stimulated by the spread of a mercantile civilization. Gradually the influence of the system began also to be felt in computation with fractions.

We can clearly see this process going on in the trigonometric tables of the Nuremberg astronomer and mathematician Regi-

omontanus, who died in 1476, and whose books and tables were standard even in the days of Stevin. In those days, and also much later until the eighteenth century, the sines, as well as the tangents and the other trigonometric entities, were conceived as lines and not as ratios, so that they were expressed in terms of a circle radius R of given length. In the sine tables of Regiomontanus we find $R = 60{,}000$, so that the sine of 30° is 30,000; later he used $R = 6{,}000{,}000$. Both data show the influence of the sexagesimal system. (Ptolemy's tables are based on $R = 60$.) The different sines are thus expressed as integers. But Regiomontanus also had a tangent table in which the tangent of 45° is 100,000, so that here $R = 10^5$, and he had another table with $R = 10^7$. The decimal system thus became established as the base for the computation of trigonometric tables. The great tables of Rhaeticus (1551), which contain to seven decimal places the values of all six trigonometric functions for angles ascending by 10″ intervals, are based on $R = 10^7$ [6].

These tables contain no decimal fractions in the strict sense of the word. Actual fractions based on powers of 10 as denominator occur in some of the many books on arithmetic which appeared during the sixteenth century. For instance, we find, in a book of 1530 written by the widely read German teacher of arithmetic, Christopher Rudolff, a table for compound interest, in which the values of $375 \times (1+5/100)^n$ for $n = 1, 2, \cdots, 10$ are written in a form which differs from our present notation only by the use of a vertical dash instead of a point as decimal separatrix, e.g., 413|4375 for $n = 2$ [7]. There are several similar cases, but none of these authors used decimal fractions consistently, and where they used them they varied their notation.

Stevin was the first in the Occident to divest the decimal fraction of its casual character. Appealing to the learned as well as the practical man, to the teacher as well as the merchant and the wine gauger, he advertised the advantages of his notation as "teaching how to perform with an ease, unheard of, all computations necessary between men by integers without fractions." In doing this, he used a notation which reminds us strongly of the sexagesimal one; where we write 47.58, he wrote 47⓪, 5①, 8②, where the unit ⓪ is called "commencement," the tenth ① is called "prime," the hundredth ② is called "second," etc. When we write 27.847 + 37.675 + 875.782 = 941.304, Stevin wrote

	⓪	①	②	③
	2	7	8	4	7
	3	7	6	7	5
8	7	5	7	8	2
9	4	1	3	0	4

In a similar way he deals with subtraction, multiplication, and division—all performed "without fractions." He ends his pamphlet with a plea to introduce the decimal system also in the measurement of lengths, areas, volumes, etc.

Stevin had the correct idea, but his notation seems clumsy to us and less elegant than that which Rudolff used half a century earlier. The circle notation was taken from the Italian mathematician Bombelli, who had used a similar notation in his *Algebra* of 1572 for the powers of the variable (anticipating our exponents in x^1, x^2, x^3, etc.). He might have used the sexagesimal notation in the form 47°5′8″, for our 47.58, as some of his followers did; he himself changed his notation occasionally and wrote 732③ for our 7.32. His notation may have had some advantages for inexperienced pupils, since the circle notation allows intermediate steps: 7⓪5①8② plus 4⓪7①5② is equal to 11⓪12①13②, which reduces to 11⓪13①3②, and this again to 12⓪3①3②. Stevin could also do away with zeros: 2②7④ means .00207. But the notation remained unwieldy, and Stevin's work would not have had its lasting influence if it had not been for Napier and his logarithms.

3

Logarithms, invented by John Napier, the Scottish nobleman, were first presented in his Latin *Descriptio* of 1614. These first logarithms are not the logarithms that we use, but are certain numbers defined with the aid of sines, which in Napier's exposition were based on $R = 10^7$. This edition of 1614 had no decimal fractions. These appear in a 1616 English translation of the *Descriptio*, with a point as decimal separatrix. This notation was adopted by Napier in his Latin *Rabdologia* of 1617, the book in which he showed how to perform computations with his "rods," the so-called "rods of Napier." Here Napier quotes Stevin's *Arithmetica Decimalis* and proposes the notation 1993,273 (with point or comma) for 1993 273/1000, although he also uses 821, 2'5" for 821 25/100. Then, in the posthumous *Constructio* of 1619, the notation has become consistent: "whatever is written after the period is a fraction." Thus 25.803 means 25 803/1000.

The great tables of logarithms, based on 10, which now appear, take the decimal notation of fractions for granted, with dot or period. With tables of such logarithms, in which the decimal part of such numbers as 43, 430, 4300 is the same, the decimal notation of fractions is only natural. Henry Briggs, in his tables of 1624, and Adrian Vlacq, in his tables of 1627, use this notation consistently, and from then on the decimal fractions with comma or dot as separatrix were generally accepted, at any rate in computations with logarithms.

Stevin's, Napier's, and Briggs' contributions were combined in two Dutch books by the surveyor Ezechiel De Decker, entitled *The New Arithmetic, First Part* and *Second Part* [8] (1626, 1627). Here we find together Stevin's *Thiende*, Vlacq's translation of the *Rabdologia*, and the Briggsian logarithms of all integers from 1 to 100,000. These two books by De Decker are a kind of glorification of the triumph of the decimal system. They stress three essential aspects of this victory: the Hindu-Arabic notation with the modern digits, the decimal fractions, and the logarithms to base 10. One change was still due, although it was already implied in the whole framework of the system, namely the rewriting of the trigonometric tables to a unit $R = 1$, a thing that even Stevin did not do. The systematic introduction of this unit $R = 1$ had to wait until Leonard Euler's *Introductio in Analysin Infinitorum* of 1748, and from that time on trigonometric entities were no longer considered as line segments but as dimensionless ratios.

4

The triumph of the decimal fractions through the work of Stevin and Napier did not mean that notation and use became immediately standardized. There were loyal Stevin followers who preferred his notation, or a slight modification of it. As late as 1739 we find the Abbé Deidier teaching that decimal fractions should be written as $89 \cdot 5^{\text{I}}\ 2^{\text{II}}\ 7^{\text{III}}\ 6^{\text{IV}}$ or 895276^{IV}; he used, however, the ordinary point notation for logarithms. Many such inconsistencies remained, again until Euler, in his *Introductio* of 1748, standardized our present notations.

The use of the decimal notation for weights and measures, also proposed by Stevin, had to wait in the West until the French revolution introduced meters, ares, and liters, and also standardized the monetary system [9]. We know that this was only partially accepted in England and in the United States, though the United States introduced a decimal monetary system as a result of the efforts of Robert Morris, Thomas Jefferson, and Alexander Hamilton [10]. As to angular measurement, the struggle is still on, and when we use our ordinary notation of degrees, minutes, and seconds, we pay our respects to a system which can boast of an age of five thousand years, certainly one of the longest ages in our whole scientific heritage.

NOTES

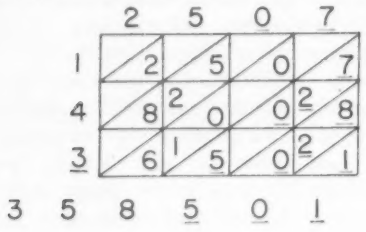
1. *The Principal Works of Simon Stevin.* Amsterdam: Swets and Zeitlinger. Vol. I, *Mechanics*, edited by E. J. Dyksterhuis, v+617 pp. (1955); Vol. II, *Mathematics*, edited by D. J. Struik, in two parts, v+976 pp. (1958).

2. Albert Girard (1595–1632) is best known as the author of the *Invention nouvelle en algebre* (1629, republished 1884), in which we find the theorem that an algebraic equation of degree n has n roots (Girard's formulation is different).

3. There have been two facsimile editions of *The Tenth* before 1958, one by H. Bosmans of the Dutch edition of 1585 (La Haye: Anvers, 1924), and one by G. Sarton of the French edition of 1585 (*Isis* 33 (1935)).

4. J. Needham, *Science and Civilization in China*, vol. III. Cambridge: Cambridge University Press, 1959. The first section of this standard work deals with mathematics.

5. *D. G. Al-Kaši Ključ Arifmetiki, Traktat ob Okružnosti*, translated and edited by B. A. Rozenfel'd (Moscow, 1956), especially p. 62; Y. Mikami, *The Development of Mathematics in China and Japan* (Leipzig, 1913), especially p. 26. This is the way in which Al-Kashi multiplies 25.07 by 14.3 to get 358.501:



(The integers are in black and the fractional parts in red. The digits are Arabic ones, which differ from our present digits, the latter being in use only since the time of the European Renaissance.)

6. These tables were expanded by Valentin Otho into the *Opus Palatinum* of 1596, a classic among the tables which has many values in ten decimals and the sines up to 15 decimals.

7. *Exempel Buechlin Rechnung belangend darbey* (Augsbury, 1530). The place where decimal fractions are introduced has been more than once reproduced, e.g., in D. E. Smith, *History of Mathematics*, vol. II, p. 241.

8. *Eerste Deel van de Nieuwe Telkonst* (Gouda, 1626); *Tweede Deel van de Nieuwe Telkonst* (Gouda, 1627). The second part was always known, but the first part disappeared until it was rediscovered by M. van Haaften in 1920; see *Nieuw Archief voor Wiskunde*, 15 (1928), pp. 49–54; 31 (1942), pp. 59–64. This discovery has shown that it was not Vlacq in 1628, but De Decker in 1627, who first published a complete table of logarithms.

9. The Chinese, as mentioned in the text, had a decimal system at least as early as the third century A.D.

10. The dollar with its decimal division was introduced by the Coinage Act of 1792, sponsored by Hamilton. This act was preceded by a resolution of Congress of 1785, the result of a report to the President of Congress by Robert Morris (1782), endorsed by Jefferson. See A. Nussbaum, *The History of the Dollar* (New York, 1957, viii+308 pp.), Chapter II; C. D. Hellman, "Jefferson's efforts toward the decimalization of the U. S. weights and measures," *Isis*, 16 (1931), pp. 266–314.

A brief history of the dollar mark

by Sharon Ann Murray, University of Maine, Orono, Maine

Contrary to widespread belief, our dollar mark did not originate in the combination of the letters *U* and *S*. The symbol actually has a Spanish origin.

Numerous theories, besides the above patriotically-motivated one, have been formulated to account for the origin of the dollar mark. According to one, the mark evolved from monographic abbreviations of *I H S* found on some Roman coins of the time of Nero. Again, the Portuguese symbol *cifrão*, used to designate thousands, has suggested another possible origin, because the Portuguese symbol is written exactly like our dollar mark. A noted historian presented a theory concerning the origin of the dollar mark from the stamp of the mint at Potosi in Bolivia. Still another theory concerns the Spanish silver coin known as the "pillar dollar," which was

widely used in the Spanish-American colonies during the seventeenth and eighteenth centuries. This coin has the Pillars of Hercules stamped on it, and it was supposed that the pillars were later copied into commercial documents. The pillar theory is strengthened because of a scroll or banner which was twisted about the pillars, and which could have suggested the *S* part of our dollar mark. Some theorizers have connected the dollar mark with the figure eight, and from this point of view see its origin in the Spanish dollar, which was known as a "piece of eight" because it was equivalent to eight smaller units called "reales." The aforementioned "pillar dollar" has an eight between the pillars, and the combination of the pillars and the eight has suggested to some a possible origin of our dollar mark. Consideration of the "piece of eight" also brought about, on the part of others, a theory connecting the dollar mark with a combination of the letter *p* and the numeral 8. One or more of these early theories may seem reasonable, but none was sufficiently established or authenticated by proper documentary evidence.

The American historian of mathematics, Florian Cajori, finally settled the matter of the origin of our dollar mark. With great scientific care and much documentary evidence, he traced the origin of the dollar mark to the sixteenth-century Spanish contraction *pss*, *ps*, or p^s for *pesos*. The florescent p^s evolved into **$** about 1775 by English-Americans who came into business contact with Spanish-Americans. The Spanish-Americans placed the *ps* or p^s *after* the numerals, but the English-Americans, accustomed to writing £ *before* the number of pounds, moved the abbreviation to the left of the numerals. In the supporting documentary evidence furnished by Cajori, the modern dollar mark in the making is particularly strikingly illustrated in a letter written by Oliver Pollack to George Rogers Clark in 1778; here both the p^s and the modern dollar mark appear side by side, the latter unquestionably resulting as a simplification of the former. It was some years, in fact not until the opening of the nineteenth century, before the dollar mark appeared in print.

Cajori's detailed account, with much facsimiled evidence, may be found in his *A History of Mathematical Notations*, Open Court Publishing Co., Chicago, Illinois, 1929, vol. II, pp. 15–29.

Aftermath

It worries me some: whatever'll become
Of my papers and books when I go?
They'd bring nothing-at-all at a second-hand stall—
It's a pity to leave them below.

Now Hardy's a treasure and Banach a pleasure
And the Knopps a delight for the mind.
There's Pólya und Szegö—well, *I* go where *they* go—
Couldn't bear it to leave them behind.

My Titchmarsh I'll need, also Hausdorff—agreed!
And Courant und Hilbert—oh yes!
And reprints of theses of various species
And copies of Bull. A.M.S.

If I should be early in reaching the pearly
Or tardily answer the call,
Could gracious St. Peter do anything sweeter
Than let in my bookcase and all?

—*Katharine O'Brien*

Edited by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

Proofs with a new format

by W. W. Sawyer, Mathematics Department, Wesleyan University Middletown, Connecticut

In the May 1959 issue of THE MATHEMATICS TEACHER, a proof was given of the theorem, "If $ABCD$ is a square, M the mid-point of DC, E and F points on DA and CB, such that $AE=BF$, then $ME=MF$."

The proof given contained fifteen steps. I do not want to criticize the individual teacher who submitted this article. It seems to me, as a foreigner, that American teachers delight in torturing themselves (and their pupils). For this theorem is just about obvious; someone who knew no geometry at all would immediately guess it to be true. I should be perfectly satisfied if a student turned in the following.

$$DE=a-b=CF$$
$$DM=\tfrac{1}{2}a=CM$$
$$\angle D=90^\circ=\angle C$$

Therefore

$$\triangle EDM \equiv \triangle FCM. \quad \text{(S.A.S.)}$$

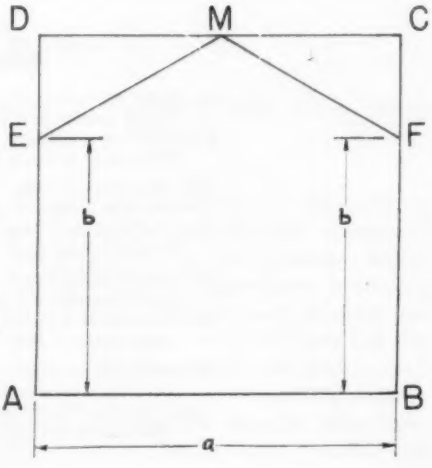
Therefore

$$EM=FM. \quad \text{Q.E.D.}$$

The drudgery of writing has here been avoided to a considerable extent. The student has written enough to show his line of thought, which is essentially sound. If you object that this is not an exercise in strict logic, *neither is the geometry text that you use in your school, nor is Euclid.* Both are full of unstated assumptions.

If you want to make geometry an exercise in strict logic, you must bring in axioms of betweenness and other refinements, and thus make geometry a topic for graduate school—incomprehensible to Grade 10. If you are not going to make it an exercise in strict logic, but rather in plausible reasoning, be honest about it.

Geometry is particularly unsuitable for teaching logic. So many of its results are so clearly suggested by the figures that



students do not see why a proof is needed; in any case, it requires profound philosophical skill to separate the logical argument from the evidence of our senses. But it is quite possible to produce reasonable arguments. The proof given above is such an argument.

One small point apart from logic: in writing a congruence, one can avoid errors by putting corresponding letters one above the other, thus

$$\triangle EDM \equiv \triangle FCM.$$

One can then draw correct conclusions, such as $EM = FM$ or $\angle E = \angle F$, and avoid incorrect conclusions, such as $EM = FC$ or $\angle E = \angle C$.

What's new?

MISCELLANEOUS

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• NEWS NOTES

Mathematicians from five countries met at Oberlin College for a briefing session this last summer prior to making a series of visits to summer institutes sponsored by the National Science Foundation. Coordinating their visits is Oberlin's Professor Wade Ellis (back row, left).

Oberlin was headquarters for the Visiting Foreign Staff Project, which brought ten mathematicians here from Europe, South America, and the Middle and Far East to participate in some 60 NSF institutes across the nation.

Visitors pictured are (front row, from left) Prof. Yasuo Akizuki, University of Kyoto, Japan; Mademoiselle Lucienne Felix, Paris, France, teacher of mathematics; Prof. Eloy Nuno de Barros-Pereira, examiner in mathematics for the Brazilian government; (back row, from left) Prof. Ellis of the Oberlin mathematics department; Prof. B. A. Amira, University of Jerusalem, Israel; Prof. George Pólya, Stanford University, a visiting lecturer for the NSF Summer Institute of Mathematics at Oberlin College; and Prof. Jorge Borbonet, University of Uruguay in Montevideo.

Also participating in the project were Prof. Luke Bunt, University of Utrecht, Holland; Prof. C. B. Panagakis, University of Athens, Greece; Mr. A. P. Rollet, England, Her Majesty's senior inspector for mathematics; and Prof. Herman Wold, University of Upsala, Sweden.

The ten visiting staff members served as observers and lecturers during the period June 21 through August 7. Direct association with scholars from abroad is expected to augment the quality of the summer institutes and promote the exchange and comparison of methods, ideas, and problems.

From Oberlin College News Bureau,
Oberlin, Ohio

• POINTS AND VIEWPOINTS

by Robert E. Pingry, University of Illinois. Urbana, Illinois

Thanks a million

A column of unofficial comment

In the May 1959 issue of THE MATHEMATICS TEACHER Dr. Henry Van Engen of the University of Wisconsin expressed his thanks for the help received in editing this journal for the last six years. The present editor is confident that he is expressing the sentiments of thousands of readers of THE MATHEMATICS TEACHER when in turn we say:

"Thanks a million to you, too, Dr. Van Engen. During your six years as editor, THE MATHEMATICS TEACHER continued to grow in quality, quantity, and influence from the high position it had already obtained under the able editorship of the previous editors. You have emphasized articles with sound mathematics content for those readers who were desirous of improving their own mathematics background; you have had articles on curriculum, teaching methods, teaching aids, history of mathematics, and tips for beginners. Teachers at all levels of training and experience have been able to find help from THE MATHEMATICS TEACHER."

The fact that THE MATHEMATICS TEACHER prints over 20,000 copies of each issue is ample evidence of the interest of the readers in the articles and departments.

The new editor enters upon this new responsibility with the realization that only through the help of able mathematics teachers who will write articles, referee manuscripts, and write suggestions to the editor can this journal maintain its present high position in these critical and exciting times for mathematics teaching. Already many persons have willingly given suggestions and much-needed advice. For this the staff is thankful and hopes that this journal will continue to have articles of sound and significant mathematical content as well as articles helpful to the mathematics teacher in the many aspects of the complicated task of teaching mathematics. You can help.

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

Arithmetic, Algebra, Logarithms and Slide Rule, Robert L. Erickson, Madison, Wisconsin, Robert L. Erickson, 1957. Paper, 11+80+13 pp., $1.50.

Literal Numbers, Equations, Functions, Robert L. Erickson, Madison, Wisconsin, Robert L. Erickson, 1957. Paper, 16+142+8 pp., $1.50.

Proportions, Variations, Progressions, Determinants, Etc., Robert L. Erickson, Madison, Wisconsin, Robert L. Erickson, 1958. Paper, 2+89+17+6, $1.50.

These are three paper-bound booklets, each of which "contains sufficient material for a semester of high school study in Mathematics," according to the preface. Although bound separately, the booklets are in actuality three parts of a single unified textbook. The booklet form was selected in order to make it lighter to carry, less bulky than a standard text, and to minimize depreciation. The purpose of the booklets is to "present instruction in that mathematics which is basic to engineering students and will be equivalent to a college course in algebra."

The author has consistently attempted to present new concepts in such a way that students with meager preparation in mathematics will not be frustrated. A rather complete review of the basic principles of arithmetic is given, with signed numbers, exponents, and logarithms discussed from the arithmetic point of view. This procedure leaves much to be desired. Inadequate definitions are the necessary outcome, since algebraic symbolism is almost completely lacking. No general laws for exponents are developed, and incorrect examples appear. On page 55, for example, $(4)^{1/2} = \pm 2$ is emphatically "defined" to be the meaning of $\sqrt{4}$. In attempting to reach the poorly prepared student, the author has completely failed to provide a sound basis for algebra or for an adequate understanding of the extended arithmetic which has been presented.

The general exposition is poor throughout. Even the grammar is not always good, as, for example, "To understand negative numbers fully, the following exercise is given" (page 32). On page 219 appears the statement: "It is impossible to obtain two equal real numbers of which the product is $\sqrt{-1}$." Definitions are poorly stated, and no attempt is made to impress the student with the importance of clear definitions.

There is a large collection of good illustrative examples and problems for solution. This is the primary redeeming factor of the booklets. The idea of having a separate booklet for each semester appears to have merit and should be considered further.—*David D. Strebe, Associate Professor of Mathematics, University of South Carolina, Columbia, South Carolina.*

Calculus with Analytic Geometry, Angus E. Taylor, Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1959. Cloth, xv+762 pp., $8.50.

This new text was written by the coauthor of a widely used calculus text. It was his intention to combine with the presentation of calculus all of the topics of a standard first course in analytic geometry, including an introduction to solid analytics. The vector approach is used for the latter. This reviewer noticed no omission from the standard content of either calculus or analytic geometry. Topics from the latter subject are introduced where they are needed in the development of calculus or where they are most appropriate. There is also review of many topics from trigonometry and algebra.

The presentation of the material maintains a high standard of rigor throughout. The student may find that certain sections, e.g., the introduction to differentials starting on p. 203, will require more than the usual number of readings. The student will find, however, that the author did his part to help in the development of a real understanding of the fundamental concepts of calculus. At the end of every second or third chapter is an extensive set of review questions and problems. The first list in each set contains questions on the concepts and definitions; the second list, entitled "Theory," contains questions calling for proofs, derivations, and statements of relationships; and a third list, present in most sets, contains problems of the more conventional type. There are plenty of regular exercises, and these include many applications to the physical sciences.

The opening section of the text contains a brief explanation, without numerical examples, of the two fundamental problems of calculus. The reviewer believes that it is more effective at this point to give the student an intuitive introduction to the derivative and integral by the use of numerical examples illustrating appropriate limits of average velocity or slope and of area. The author, however, has neither done this at the beginning nor when the deriva-

tive is first introduced using the instantaneous velocity approach.

This text uses the modern ordered-pairs definition of function. The problem of definition of the inverse function is handled nicely wherever it arises, e.g., with inverse trigonometric functions on p. 182. In the formal definition of limit, k and h are used in place of ϵ and δ. The reviewer was disappointed to find only one numerical example of the definition, and in it a perfectly general expression for h in terms of k was not obtained. The theorems on limits are assumed in the development in Chapter 1 and are later proved in Chapter 14.

The treatment of the antiderivative and definite integral seems commendable. The integral sign is not used in Chapter 2, "The Inverse of Differentiation," in which the law of the mean is proved, antiderivatives are found formally, and area is found both as the limit of sums and by antidifferentiation. The integral sign is introduced in Chapter 5 for additional formal work with antiderivatives. In the next chapter the definite integral, using the notation $I_a^b(f)$, is defined as the limit of a sum; the mean-value theorem is proved; and the fundamental relations between derivatives and integrals are obtained. It is here pointed out clearly that the result used in Chapter 2 in finding areas by antidifferentiation was essentially a geometrical interpretation of the general result.

In the reviewer's opinion, an outstanding feature of this text is the integral definition of the natural logarithm function, $L(x) = \int_1^x dt/t$, $x > 0$. Some background for the definition is given, and from the definition the basic properties of the logarithmic function are obtained. The exponential function, $E(x)$, is introduced as the inverse of $L(x)$; and its properties point the way to a satisfactory definition of a^u for irrational u, an approach which avoids the difficulties inherent in the usual development of such a definition.

This text appears to contain sufficient material for at least three semesters of work for a class meeting four times per week. It is a well-written text which should prove successful for able college students and might be used with superior high school students.—*John A. Schumaker, Montclair State College, Upper Montclair, New Jersey.*

Elementary Algebra, Donald S. Russell, Boston, Allyn and Bacon, Inc., 1959. Cloth, ix+297 pp., $4.50.

This book, designed for college freshmen, deals with the usual content of ninth-grade algebra. It is copyrighted 1959, but it is not a book for 1959. That the spirit of "modern mathematics" eludes it entirely may be shown by the following examples, which are typical:

In place of the modern emphasis on *careful definitions*, we find (p. 20): "A real number is any number that can be associated with a point on a line." Again (p. 253): "An imaginary number is the indicated square root of a negative number." Also (p. 3): "... $A = \pi r^2$ (where r equals the radius of the circle and π equals 3.1416 or 22/7) ..."

The logical structure of mathematics is nowhere apparent. Although one or two axioms are *mentioned*, they are never *used*, so that (for instance) multiplication of one polynomial by another is to be carried out by a rote rule; the student is not shown that this "rule" is in fact a *theorem* that follows from the axioms. Similarly, multiplication of signed numbers, which is capable of many elegant treatments, is badly slighted and is not presented in relation to the axioms.

Verbal rules are used in place of understandings or geometric visualizations. A case in point is the verbal rule for adding signed numbers. Similarly, Cartesian co-ordinates are deferred until so late that many opportunities to use them must be omitted.

All in all, this volume presents algebra as a one-term course in symbol-pushing. It is an interesting relic of the pre-Beberman, pre-Commission, pre-SMSG era in American mathematics education.—*Robert B. Davis, Syracuse University, Syracuse, New York.*

Geometrie als Sprache der Formen, Hermann von Baravalle, Freiburg im Breisgau, Germany, Novalis-Verlag, 1957. Cloth, 240 pp., about $8.

Geometry as the Language of Forms is a fascinating book. Numerous readers recall Dr. Baravalle's interesting article, "Geometric Drawing," in the Eighteenth Yearbook of the National Council of Teachers of Mathematics. Now his book on forms affords a rich abundance of similar materials.

Indeed, the new subject matter transcends an approach via drawing. Along with his superb figures the author carefully delineates pertinent mathematical relations. The book is useful for building concepts prior to studying geometry as theorems deduced from a set of postulates.

The book is a treasury. Unfortunately reviews of books have to be brief. Only a listing of some of the topics Professor Baravalle treats can appear here.

Triangles, quadrilaterals, circles, regular polygons, spirals, conchoids, conic sections, periodic curves, centroids, involutes, and evolutes receive a full treatment. So do the copying, enlarging, and diminishing of figures. The Pythagorean Theorem, inversion, and projective geometry appear, each in a chapter of its own.

In this work geometry proceeds from its elements, such as lines and points. Beginning with a simple equilateral triangle, for example, the author derives all kinds, and combinations, of triangles. The figures grow ever more interesting. Eventually the locus of the orthocenter of a triangle, as the apex of the triangle is moved sidewise and the base kept constant, is shown to be a parabola. Similarly for numerous other figures, the study unfolds neatly from the drawing board.

That the book appears in German need not dismay the teacher who does not read that language, for an English edition is in preparation. Besides, its 96 pages of drawings (411 figures in all) are themselves worth the price of the book.—*Irvin H. Brune, Iowa State Teachers College, Cedar Falls, Iowa.*

Foundations of Advanced Mathematics, William E. Kline, Robert Oesterle, and Leroy M. Willson, New York, American Book Co., 1959. Cloth, vi+519 pp., $4.96.

In both order and method, this book departs refreshingly and with profit from traditional presentations of much of the material it covers. It is written to, as well as for, the student, dispensing with all prefaces, acknowledgements, and notes to the teacher but including a two-page note to the student which the teacher is sure to endorse heartily. The exercises, for which no answers are given, tend to illustrate points made in the text rather than to challenge the student's ingenuity, and the text itself can be read easily by a high school student. The reader is usually warned when an argument is to be less rigorous than it might be, and facts dignified by the title "Theorem" either are proved or intuitively verified, with the absence of proof specifically noted.

The book is divided into six sections organized with a minimum of interdependence: Solid Geometry, Plane Trigonometry, Plane Analytic Geometry, Calculus of Polynomial Functions, Statistics, and Algebra, together with four-place tables of Trigonometric Functions (excluding secant, cosecant), Common Logarithms of Numbers and of the Trigonometric Functions, and Square Roots. It would be particularly suitable for independent study by a person who had completed plane geometry and three or four semesters of algebra, and could also be used as the basis for class study during the last two or three high school semesters. In the latter case, however, the teacher would probably wish to provide supplementary exercises and, if the goal of the class was to complete the work prerequisite to the calculus, considerable material on solid analytic geometry would also have to be supplied.

The sections on Solid Geometry and Trigonometry are adequate treatments of the courses they represent. The intuitive approach is stressed in the former, although 18 theorems are proved deductively, and facts to be recalled from plane geometry are generously interspersed throughout the section. Noteworthy in the Trigonometry section are: delay of logarithms and logarithmic solution of triangles until the end of the section; a particularly clear exposition accompanying trigonometric identities and equations; and the introduction of the six trigonometric ratios by defining them with reference to the coordinate axes, and later applying them to "uncoordinated" right triangles (thus minimizing the sign difficulties experienced by many students when the process is reversed). DeMoivre's Theorem is not discussed here or elsewhere in the book, and one might wish for more opportunities to gain facility with radians as arguments for the trigonometric functions, but in other respects all phases of a standard trigonometry course are covered in considerable detail.

The other four sections make no claim to cover their respective subjects, but rather seem designed to organize and amplify under a few headings the many seemingly unrelated topics of the traditional College Algebra course. Relegated to one chapter in the last section is a brief review of basic techniques usually learned in the first three semesters of algebra, after which, in the final four chapters of the book, the topics Mathematical Induction, Determinants, Complex Numbers, and Theory of Polynomial Equations are discussed. Chapters on Permutations and Selections (=Combinations), Probability, Set Theory, and Elementary Statistics comprise the section which precedes the Algebra section, and these chapters present a unified treatment of the subject matter they contain; a chapter relating probability and set theory, with emphasis on Venn diagrams, is particularly welcome.

Several topics not usually stressed in Analytic Geometry are given careful attention under this heading: real numbers and their relation to coordinate geometry, the geometric aspect of inequalities, and the concept of a function and its graph. The subject of lines is also covered extensively (including an elementary derivation of the formula for distance between a point and a line, replacing the usual discussion based on the normal form), after which locus, elementary curve-tracing, parametric and polar equations, and certain special types of higher plane curves are more briefly considered. Finally, conic sections are introduced and explored with the help of many good diagrams of the sections of a cone; translation of axes is discussed, and rules are given for determining the type of conic represented by a given unrotated equation of second degree, but techniques of rotation are not presented, and neither graphical nor algebraic solutions of simultaneous second-degree equations are considered.

The brief section on Calculus develops the derivative and antiderivative of polynomial functions. Limits and continuity are approached intuitively, although some limit theorems are stated without proof, and the student who had previously studied the first section of the book would have there gained some appreciation of limits from a geometric viewpoint. The Δ-process is well-explained, and the advantages gained from using properties of the derivative in graphing polynomials are also clearly illustrated and discussed.

Throughout the book, the format is excellent: illustrations are clear, and important facts, new terms, and section headings have been

emphasized through bold-faced type and italics. In every way the book seems well-suited to the task of encouraging a student to learn more about mathematics.—*Mildred J. Brannon, Urbana High School, Urbana, Illinois.*

Foundations of Mathematics, Carl H. Denbow and Victor Goedicke, New York, Harper & Brothers, 1959. Cloth, xviii+620 pp., $6.00.

This book is intended for a year's work at the college freshman level, although parts of it could be used in high schools. Presumably it is intended as preparation for an intensive course in analytic geometry and calculus, but it could easily be used in a terminal or "cultural" course. For students of modest background the book should prove stimulating and worth while. Students with a strong high school background perhaps should proceed at once with an intensive course in calculus.

There is a variety of material discussed. Besides the topics one normally expects in freshman texts, there are chapters on groups, number systems, logic and sets, probability and statistics, and number theory. Chapter 14 on calculus includes the fundamental theorem.

The significant feature of the book is its flavor. The authors are more concerned with bringing the student to realize the nature of mathematical systems than in developing great technical dexterity. It is difficult to judge the success they have achieved without actually actually teaching the book. Certainly the book should remove from a student's mind the "mystery" of certain concepts, and the informal style lessens the austerity inherent in mathematics. It seems likely that an able teacher could use the book very effectively to generate real interest in mathematics.

The preface describes well the authors' aims. Because mathematics to the man in the street means "a set of logical relations plus their relation to the real world," the authors try to do more than formally develop mathematical systems. Hence they often present mathematics as an "experimental science." For example, the chapter on group theory begins with an entertaining story concerning a man who permutes the living room furniture, and eventually understands the permutation group on three letters. The authors are adept at motivation. Chapters 1–4 constitute a preliminary survey which reviews familiar ideas "in a setting to pave the way for more mature approaches." There are numerous historical references.—*Merrill Shanks, Purdue University, W. Lafayette, Indiana.*

Freshman Mathematics, Paul K. Rees, Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1959. Cloth, x+285 pp., $4.95.

From the preface: ". . . primarily for prospective elementary teachers and non-science majors in an arts college; however, it can be used by those students who need a refresher course in algebra or who must study algebra as a background for work in the mathematics of finance."

The choice of topics was made in a unique manner. Professor Rees used the results of a study in which authorities in the field were asked to indicate the value of the topics in a list selected from textbooks used in a general mathematics course. The topics finally included fall into the following groups: algebra, trigonometry, arithmetic, the straight line, compound interest, annuities, statistics, and probability. The first three comprise about 75% of the book.

A very faithful attempt was made to arrange the exercises in each list in groups of four of nearly the same degree of difficulty, resulting in convenience for the instructor in making assignments.

The material in this text is presented and developed in the traditional intermediate algebra style. Topics are presented with very little theory; examples are used that fit precisely this presentation; then exercises are tailored very closely to the examples.

In my opinion the students for whom this book was designed need more than this. Without saying so, this book is remedial. Prospective elementary teachers or liberal arts students will use it. They need to know why things work in mathematics; they need to see how mathematics holds together in an interrelated logical structure. In these aspects this text falls short. Fractions are presented from the standpoint of operations with them—no suggestion is given of the meanings of fractions. Rounding off decimal fractions is given in a set of rules telling how, but not why. In chapter 8, interpolation is presented. Here an opportunity to tie in previous material on graphs and linear equations was not utilized. Of course, some of these matters are decided by taste and personal preference. Yet there are too many instances of sacrificing "why" to "how" for this to be dismissed on those grounds.—*Lyman C. Peck, Ohio Wesleyan University, Delaware, Ohio.*

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan and William C. Lowry, University of Virginia, Charlottesville, Virginia

Helping pupils develop their ideas

by William C. Lowry, University of Virginia, Charlottesville, Virginia

Jack, a seventh grader, is excited. "Miss Jones, I've found a new way to get the average of two numbers. Take 12 and 18, for example. Subtract 12 from 18; that's 6. Divide the difference by 2; you get 3. Add 3 to 12, and you get the average, 15."

Teachers vary widely in their reactions to comments such as these. For purposes of discussion, let us consider three different teachers and their reactions. Such teachers are by no means absent from our classrooms.

Miss A's reaction is as follows: "Yes, Jack, 15 is the average of 12 and 18; but you know that is not the way we find averages. We add the two numbers and divide by 2. Stick to the correct way and you won't be making mistakes when you are asked to find averages."

We shall be charitable to Miss A. Arithmetic may not be her forte. She may operate in and teach arithmetic entirely by rules, with little comprehension as to why the rules work. She does not know Jack's method; she has never seen it before; and she does not feel competent to test it herself. She must, of necessity, stick with the "tried and true." Miss A may have learned arithmetic as a body of facts, skills, and techniques for which the teacher showed the "what" and "how," and the pupil learned the "what" and "how" by practicing that which the teacher had shown. Miss A is of the opinion that it is not a good policy to let the pupil experiment. In fact, arithmetic, as far as she is concerned, does not lend itself to such a method of teaching and learning.

Jack's reaction to Miss A's remarks may be, "I'll show her; I *know* it will work." If that is his reaction, he will probably try his method on several pairs of numbers until he assures himself that it is correct; or he may, after becoming fairly competent with his method, present it to someone who will analyze it with him. Not all is lost with Miss A, at least, not all for some pupils. In fact, it may be one way of challenging them, as Jack was challenged to "show her." Often, however, the pupil's reaction is one of chagrin or even anger toward the teacher who so callously dismisses his "find"; and his desire to prove his method is motivated by a feeling of revenge. Needless to say, such motivation is not the most desirable kind.

Miss A's reaction, though it leaves much to be desired, is in many respects superior to that of Miss B. Miss B kills the spirit of inquiry, the eternal wonder of youth. Her answer is, "Yes, Jack, your method works. In fact, it can be generalized to work for more than two numbers. It is nothing new. Some day you will learn that method; but right now I want you to find averages by the method we are using in class. Add the numbers and divide by the number of numbers. Learn that well,

be careful and neat, show all your work, and you will get a good grade."

Not only must Jack find averages by the prescribed method even though there are other methods, one of which he has found himself, but he is doing it for a grade. The excitement of learning and discovering things for himself is of little consequence, according to Miss B. There is nothing new or exciting about what he has found; it has all been done before. And, in truth, the exciting thing is not *what* Jack discovered; it is that Jack made the discovery himself. The teacher should put the emphasis where it belongs: "How smart of you, Jack, to have discovered this!" There is only one thing Miss B said that Jack may react to constructively, and do something about: "In fact, it can be generalized to work for more than two numbers."

Notice that Jack arrived at his method on his own. He was not prodded or "led into" his initial inquiry and his discovery by the teacher. While this is not meant to discount the importance of the discovery method when led by the teacher, a teacher should make the most of ideas which arise as Jack's did, initiated solely by the pupil and unaided by teacher or textbook suggestions.

Miss C's reaction is of a different kind. "Are you certain your method will work for other numbers, Jack? You picked two special numbers. It may be that it will work only for those two. Have you tried odd numbers? Will it work for them?"

Jack: "Let's see. Take 13 and 19; the difference is 6. Half of that is 3, and 13 plus 3 is 16. And 16 *is* the average of 13 and 19. Now I'll try an odd number and an even number, say 23 and 28. The difference is 5; half of 5 is $2\frac{1}{2}$. Twenty-three plus $2\frac{1}{2}$ is $25\frac{1}{2}$; and that is the average. Yes, I think it will work for any two numbers."

Miss C asks, "Have you tried 3257 and 4, or $7\frac{1}{2}$ and $11\frac{3}{4}$?"

Jack is sure his method will work, but he takes paper and pencil and after some jotting of figures, reiterates his belief in his method. And let us not misunderstand; Jack is sure his method will work. Nevertheless, Miss C suggests, "But, Jack, you haven't tried all the pairs of numbers possible, have you? It may fail on the next two you try. Suppose someone asks you to show him that your method works for any two numbers. Are you going to try to prove your method by trying it on all possible pairs of numbers, or will you try to give reasons why it will work?"

Jack answers somewhat as follows: "It has to work, because if you count halfway down from the larger number to the smaller number or halfway up from the smaller number to the larger, you will get the average. (This is true for two numbers, of course, even though Jack is describing how to find the median, not the arithmetic mean.) Or you could look at it this way. If you find the average of two numbers which are the same, such as 16 and 16, the average is that number, 16, because you simply double the number and then divide that answer by two, getting back to the original number. Now take 16 and any other number, such as 20. Since the average of 16 and 16 is 16, then the average of 16 and 20 will be 16 plus half of how much greater 20 is than 16."

This is close enough for the moment, so Miss C asks, "Jack, do you think your method will work for finding the average of more than two numbers? For example, will it work for three numbers?"

Jack starts, "Let's try 12, 15, and 18. Since there are three numbers, I'll take one-third of 12, that's 4; one-third of 15 is 5; and one-third of 18 is 6. Four plus 5 plus 6 is 15, and 15 is the average."

Miss C permits Jack to finish his computation using this different method before she questions, "Is that the method you were using for two numbers? Did you take half of 12 and half of 18 and add the answers together to get the average, 15?"

Jack admits that he didn't. "I took the difference and divided by 2 . . . . Now, if I do that, I have 15 less 12 is 3 and 18 less 15 is 3. Three plus 3 is 6, divided by 3 is 2;

and 12 plus 2 is 14. It doesn't work; the average is 15."

Miss C recognizes some help is necessary. "Maybe you took the wrong differences. Let 12 be one of your numbers each time you subract."

Jack, after a little silent mental calculation, concludes that it works and then proceeds to try it on another triplet of numbers and then on four numbers. Miss C commends him for his work, but more can be done. "Jack, let us return to the numbers 12, 15, and 18. Take the differences between them and some other number, such as 7, and use the same method."

Jack calculates aloud, "Twelve less 7 is 5; 15 less 7 is 8; and 18 less 7 is 11. The sum is 24, and 24 divided by 3 is 8. Seven plus 8 is 15. It works!" Pleased with the new "find" Jack tries using 10, then 5, then 14. "Fourteen minus 12 is 2; 15 minus 14 is 1; 18 minus 14 is 4. The sum is 7, and 7 divided by 2 is. . . . Wait, this isn't going to work. I'm getting a fraction."

Miss C knows some help is necessary here, since Jack has had only an introduction to directed numbers. "Jack, do you subtract 12 from 14 or 14 from 12? When you subtract 14 from 12, remember the answer is negative 2."

"Oh, yes. The sum of 4 and 1 is 5, less 2 is 3. One-third of 3 is 1, and 14 plus 1 is 15."

Miss C now asks Jack to review his method, to describe what he did. He describes it somewhat as follows: "If I am finding the average of three numbers, I choose any number, not necessarily one of the three. I subtract this number from each of the three numbers, paying attention to the differences which are negative. I add the positive differences and subtract the negative differences, divide by three, add that answer to the number I chose at first, and get the average.

"Very good, Jack. You have just worked out a method for finding averages that is known as the guessed-mean method. The number you choose arbitrarily is a guess as to what the average is. Then your method of finding differences between your guess and the given numbers is used to correct your guess. But now let us return for a moment to the other way you used to find the average of 12, 15, and 18. You took one-third of each number separately and added those numbers to find the average. Do you think you could use that method in all cases? Can you think of a case in which it will not work?"

Jack is certain it will always work. "In the usual method we add all three numbers and then take one-third of the sum. We should be able to take one-third of each number and then add. Certainly, we can. That is the same as the method we used to find one-fourth of 84 by finding one-fourth of 80, then one-fourth of 4, to get 20 and 1, or 21."

Not all of the above account is fictitious. The "original" method for finding the average of two numbers was proposed to the writer by his twelve-year-old son late one night during a return trip from Lexington, Virginia to Charlottesville, Virginia, a distance of some 60 miles. The questions asked by "Miss C" and the answers given by "Jack" constitute a part of the examples tried during that drive. The discussion does not indicate some hesitations on the part of the computer, nor does it include all of the triais made. Those that are given, along with the errors made, are described essentially as they occurred during that 60 miles. Teachers can get even better mileage with this question-answer technique in helping pupils to develop their ideas in a normal classroom situation.

• NOTES FROM THE WASHINGTON OFFICE

Edited by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D.C.

Registrations at NCTM conventions

Below are registration reports for three recent conventions of the Council. Our conventions continue to set attendance records. The registrations of 636 and 2286 participants respectively for the Eighteenth Summer Meeting and Eighteenth Christmas Meeting were the largest, for meetings of these types, in the history of the Council. The total of 1778 registrants for the Thirty-Seventh Annual Meeting made this the second-largest annual meeting. The location of this meeting away from our midwest concentration of members was undoubtedly the sole reason why it, too, did not set an attendance record.

These attendance records seem to be the result of two factors: (1) the growing interest, on the part of both teachers of mathematics and the public, in the improvement of the teaching of mathematics, and (2) the growing awareness among teachers of mathematics of the contribution that the NCTM is making toward bringing about this improvement.

Registrations at the Eighteenth Summer Meeting

The National Council of Teachers of Mathematics, Colorado State College, Greeley, Colorado, August 17–20, 1958

Arizona	11	Nebraska	14
Arkansas	2	New Hampshire	2
California	48	New Jersey	8
Colorado	138	New Mexico	26
Connecticut	3	New York	10
Delaware	2	North Carolina	3
District of Columbia	5	North Dakota	1
Florida	2	Ohio	13
Georgia	3	Oklahoma	13
Idaho	1	Oregon	10
Illinois	62	Pennsylvania	14
Indiana	19	South Dakota	2
Iowa	28	Tennessee	6
Kansas	28	Texas	42
Kentucky	1	Utah	4
Louisiana	3	Virginia	2
Maine	1	Washington	7
Maryland	1	Wisconsin	16
Massachusetts	4	Wyoming	11
Michigan	26	Hawaii	4
Minnesota	14	Canada	4
Mississippi	1		
Missouri	13		
Montana	8	Total	636

Registrations at the Eighteenth Christmas Meeting

The National Council of Teachers of Mathematics, Sheraton-McAlpin Hotel, New York, New York, December 28–30, 1958

Alabama	3
Arkansas	1
California	7
Colorado	6
Connecticut	108
Delaware	21
District of Columbia	33
Florida	9
Georgia	4
Illinois	73
Indiana	21
Iowa	16
Kansas	8
Kentucky	6
Maine	21
Maryland	60
Massachusetts	136
Michigan	27
Minnesota	9
Mississippi	4
Missouri	5
Montana	2
Nebraska	4
New Hampshire	22
New Jersey	373
New York	868
North Carolina	7
Ohio	57
Oregon	3
Pennsylvania	249
Rhode Island	20
South Carolina	5
Tennessee	6
Texas	12
Vermont	15
Virginia	23
Washington	3
West Virginia	13
Wisconsin	8
Wyoming	1
Canada	14
Foreign	2
Puerto Rico	1
Total	2286

Registrations at the Thirty-seventh Annual Meeting

The National Council of Teachers of Mathematics, Baker Hotel, Dallas, Texas, April 1–4, 1959

Alabama	6
Arizona	14
Arkansas	17
California	34
Colorado	12
Connecticut	5
Delaware	2
District of Columbia	14
Florida	17
Georgia	1
Illinois	83
Indiana	27
Iowa	12
Kansas	14
Kentucky	1
Louisiana	37
Maryland	7
Massachusetts	10
Michigan	27
Minnesota	17
Mississippi	2
Missouri	15
Nebraska	9
New Hampshire	1
New Jersey	14
New Mexico	5
New York	28
North Carolina	1
North Dakota	1
Ohio	17
Oklahoma	69
Oregon	2
Pennsylvania	12
South Carolina	1
South Dakota	1
Tennessee	12
Texas	1182
Utah	3
Virginia	9
Washington	3
Wisconsin	12
Wyoming	1
Canada	19
Foreign	2
Total	1778

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Minutes of the Annual Business Meeting

Baker Hotel, Dallas, Texas

April 3, 1959

Dr. Harold P. Fawcett, President, called the meeting to order at 4:10 P.M.

I. A motion was made, seconded, and passed to approve the minutes of the meeting of April 11, 1958, as printed in the *Journals*.

II. *Report of the Recording Secretary.*

Dr. Houston T. Karnes, Recording Secretary, gave a brief review of the actions of the Board during the past year. This review included items which were thought to be of interest to the members. In his opening remarks, Dr. Karnes stated that it was impossible, in a brief time, to give a complete summary of the Board meetings. He pointed out that a large share of the meetings are devoted to committee reports and the discussion of matters before decisions are reached. He exhibited a folder of the committee reports, which were submitted at the Board meeting immediately preceding the Dallas convention, and indicated that the folder of reports would form a book of considerable size. The review follows:

A. Immediately following the Cleveland meeting, the Board authorized the printing of a new Directory. This, of course, has been done and is a distinct aid to many people.

B. At the New York meeting, Dr. Robert E. Pingry, of the University of Illinois, was elected to a three-year term as Editor of THE MATHEMATICS TEACHER to succeed Dr. Henry Van Engen, who has served in this capacity for six years—the maximum time allowed under our Bylaws.

C. Dr. Irvin Brune has been authorized to supervise the writing of a Guidance pamphlet to replace the one now in existence.

D. Future meetings of the Council:

	Spring	*Summer*	*Christmas*
1959		Ann Arbor	
1960	Buffalo	Salt Lake City	Tempe, Ariz.
1961	Chicago	Toronto	
1962	San Francisco	Madison	
1963	Pittsburgh	Boulder	

In addition, the Board approved a one-half day meeting in conjunction with the NEA St. Louis meeting this summer. Also, a full-day meeting in conjunction with the Chicago meeting of the M.A. of A. next January. This gives us four meetings for 1960.

The Board decided to schedule no future Christmas meetings as a regular order, but to consider any invitations which might be extended.

E. Future yearbooks:
1. The 25th Yearbook, *Instruction in Arithmetic*, will be released in the Spring of 1960;
2. The 26th Yearbook, *The Evaluation of Achievement in Mathematics*, will be released in the Spring of 1961.

F. In view of increased costs of a convention, the Board has increased the registration fee for members from fifty cents to $1.00, and for nonmembers from $1.00 to $3.00. This change will become effective with the 1960 Buffalo meeting.

G. The anticipated report of the Secondary School Curriculum Committee will appear in the May issue of THE MATHEMATICS TEACHER. Reprints will be available.

H. The Board has increased the number of issues of *The Arithmetic Teacher* from six to eight per year.

I. In view of the steadily increasing duties of the President, the Board has given this office much relief by releasing the President from Convention Program responsibilities. For all future meetings, the program will be the responsibility of a program committee composed of the four vice-presidents.

J. Budget:
Time does not permit a detailed report of the new budget at this session. The budget for 1959–60, as adopted by the Board, will, however, be published in the October issue of our *Journals*. It is thought well to report on three items today.

1. The operation of the Council is steadily growing and has reached such proportions that the Board has approved an operational budget of approximately $195,000 for 1959–60. We are thankful that the income permits this to be done. This means increased and better service to the membership and increased activities in the field of mathematics education. It should be said that the budget is not set at this figure because the money is available through an increasing membership, but rather an increasing membership and increasing demands due to the present impetus on mathematics makes a budget of this size necessary. If the money were not available, through dues, it would have to be transferred from our reserve security and thus deplete this fund in one year or from the publications account and thus greatly decrease our effectiveness in this area.
2. In view of the increasing demands on the Washington Office and especially upon the Executive Secretary, it is found necessary to add a professional assistant to our staff.
3. Because of the increasing amount of editorial work in the Washington Office and because the NEA finds it necessary to curtail the services of its editorial department, the Board deemed it necessary to add an editorial assistant to the Washington Office.

III. *Report of the Executive Secretary.*

Mr. M. H. Ahrendt, Executive Secretary, gave the following brief report.

A. The present staff of the Washington Office consists of the Executive Secretary and twelve secretarial and clerical persons. In December, the Office was moved to larger quarters in the new NEA Building. Mr. Ahrendt indicated that these quarters would no doubt be somewhat crowded by this coming fall in view of the manner in which the work was growing.

B. On February 1, 1959, the total membership and subscriptions was 22,215. This was an increase of 4,000 over the same date of the previous year. At the present rate, this total should reach 23,500 or 24,000 by May 1. When most organizations increase dues, there is usually a decline in membership. This, however, did not happen in the case of the Council.

C. In order to further show the growth of the Council, Mr. Ahrendt gave the printing orders for the April, 1959 issue of our *Journals*. The printing order for THE MATHEMATICS TEACHER was 20,500; for *The Arithmetic Teacher*, 9,500; and for *The Mathematics Student Journal*, 80,000.

D. Mr. Ahrendt reported that the financial condition of the Council was excellent, with income ahead of budgetary estimates. A complete financial statement will appear in the October issue of the *Journals*.

E. Mr. Ahrendt stated that the addition of the addressograph machinery was not only a help in the Washington Office, but also an aid to other professional organizations which had purchased the services of this equipment. To date, the machinery has been used for all or part of the list for thirty-one separate mailings.

F. In addition to the membership listing in the Washington Office, a new registry has been developed. This is The U. S. Registry of Mathematics and Science Teachers. This registry was developed in co-operation with the National Science Teachers Association and subsidized by the National Science Foundation. At the present, it contains the names of 120,000 persons, of which 80,000 teach at least one course in mathematics.

G. New publications:

In addition to the *24th Yearbook* and several supplementary publications, there are three publications which should be of interest to the teachers of mathematics. They are:

1. Career pamphlet
   This pamphlet was produced with the co-operation of NSF-NRC. All members of the Council should receive a copy of this publication before too long.
2. Articles by George Boehm in *Fortune Magazine*
   Reprints of these articles are being assembled and all members of the Council should receive a copy soon.
3. *Mathematics for the Academically Talented Student*
   This publication was developed in co-operation with the NEA project on the academically talented student. Copies of this publication are available in the Washington Office.

IV. *Report of the President.*

President Fawcett stated that a report of his activities and the over-all work of the Council would extend this session much beyond its scheduled closing time. He stated further that the reports being given by others at this meeting would give the membership a good over-all picture of the work of the Council during the past year.

President Fawcett said that, under the Bylaws, the President has two duties. They are:

1. the usual responsibilities associated with the office of President, and
2. to preside at the Board Meetings.

He remarked that it did not take him long to find that these reponsibilities were great and time-consuming. He thanked the membership for the opportunity that had been given him, and said that he had developed two new respects for the Council because of:

1. the effect that the Council had had upon his own life, and

2. the respect with which the Council is held in professional circles.

The work of the Council, President Fawcett stated, was carried on through an overwhelming committee structure. At the present, there are some twenty-eight committees, containing a personnel of 139 people. The work of these committees delves into all phases of mathematics education. From the work of these committees, important developments are taking place, which will be of great aid to the teachers of mathematics.

President Fawcett extended his thanks and appreciation in general to the membership for the excellent co-operation which he had received during the past year, and more specifically to the work of the members of the Board, the editors, and to the members of the various committees. In addition, he expressed his thanks and appreciation to the four vice-presidents and to Former-President Mrs. Marie S. Wilcox, for the excellent work they did in assisting with developing the programs for the meetings of the Council.

V. *Problem-Policy Committee.*

This past winter, on the recommendation of the President, the Board authorized the establishment of a new committee to be known as the Problem-Policy Committee. In introducing the chairman of this committee, President Fawcett made the following statement:

> According to the Bylaws adopted March 29, 1949 and amended April 24, 1954, the purpose of the Council is "to assist in promoting the interests of mathematics in America, especially in the elementary and secondary fields." Since its creation thirty-nine years ago, it has steadfastly served this high purpose and throughout this period has continued to emphasize the importance of mathematics in a program of American education. Until recently it has, in fact, been the only professional organization dedicated to the improvement of mathematics teaching on both the elementary and secondary levels. Within recent years, however, there has been a decided change in the intellectual and academic climate of America, and today the improvement of mathematics education from kindergarten through college is a problem which has commanded the active interest of other professional organizations and is of concern to all thoughtful people. Significant curricular studies are in process. New instructional materials are being developed. An in-service program of large proportions is now operating, and the collegiate program for the mathematics teachers of tomorrow is under revision. Large financial assistance is available for all of the projects, and it is evident that the original purpose of the Council, "to assist in promoting the interest of mathematics in America, especially in the elementary and secondary fields," is now receiving widespread support. What then should be the role of the Council in this changed intellectual climate?
>
> To study this and other policy problems, I have created, with the approval of the Board of Directors, a "Problem-Policy Committee" with responsibility for keeping alert to movements in mathematics and in mathematics education, both at home and abroad, for defining areas of genuine concern to the Council, and for proposing to the Board those policies which will best serve the interest of mathematics education. The committee, consisting of Phillip Jones, John Mayor, Philip Peak, and Howard Fehr, Chairman, has had one meeting, and at this time it is appropriate to have a preliminary report from the chairman, Howard Fehr.

VI. *Report of the Problem-Policy Committee.*

Dr. Fehr gave his report verbally, using the following outline:

A. Origin: Events occurring during the last six years.
   1. Upsurgence of international need for mathematics. Impetus in mathematics education
   2. Formational support of other groups
   3. Relation to other groups in our services to the membership of the Council

B. What are the problems of the Council and what should be its policy?
   1. Teacher education: regional conferences
   2. Publications: monographs for teachers; yearbooks for parents and laymen
   3. Annual conferences
   4. Adult education program
   5. Teacher certification and teaching standards
   6. Role of television
   7. Elementary school

C. As a result of the deliberation on the topics under A and B above, the Committee recommended that a meeting composed of some twenty-odd participants be held as early as possible this fall, to deliberate upon the future work of the Council in view of the wide range of activity in the field of mathematics today. The Board adopted this recommendation and authorized the President to proceed with plans for the meeting. From the results of the meeting, it is expected that a well-defined program and policy would be adopted for the work of the Council during the next several years.

VII. *Report of the 1959 Committee on Nominations and Elections.*

Dr. Milton W. Beckmann, chairman, gave the report for the committee. He stated that, in addition to a meeting of the Committee in Cleveland last Spring, the Committee met for a day and a half in Chicago in the early part of the Summer. During this meeting, careful consideration was given to the many recommendations which had been received. As a result of this meeting, the following were nominated:

1. Vice-President for Colleges
   Phillip S. Jones*
   Z. L. Loflin
2. Vice-President for Junior High Schools
   Marian C. Cliffe
   Mildred B. Cole*
3. Directors
   Carol V. McCammon
   Philip Peak*
   Christine Poindexter
   Irene Sauble
   Oscar F. Schaaf*
   Henry Van Engen*

Those names marked by an asterisk are the persons who were duly elected.

Dr. Beckmann stated that a total of 3,747 votes was cast. He went on to say that it was regrettable that such a small percentage of the 15,000-odd membership took part in the voting. In closing, he suggested that more interest should be generated in the election of officers.

VIII. *Report of the Resolutions Committee.*

The Resolutions Committee was composed of Miss Alice M. Hach and Dr. H. Vernon Price. Miss Hach presented the following report.

The National Council of Teachers of Mathematics depends upon loyal members and supporters for its existence. Many people devote a great deal of energy and time to making a convention a success. It is recognized that a large part of this effort is contributed by the local committees. It is proper, therefore, that the National Council express its appreciation to the Greater Dallas Council of Teachers of Mathematics and the Dallas Independent School District.

Acting as host for a convention of this size had been a major undertaking. But you in Texas have been noted for doing things on a grand scale. Your hospitality and your contribution to the success of this thirty-seventh annual meeting have been in keeping with this tradition.

Following the reading of this resolution, Miss Hach moved its adoption. The motion was duly seconded and passed unanimously.

IX. Dr. H. C. Christofferson, Former President, moved that the President and the Board be extended a vote of thanks and complimented highly on the outstanding work that was accomplished during the past year. This motion was duly seconded and passed unanimously.

X. Dr. Donovan A. Johnson moved that Dr. Henry Van Engen, retiring editor of THE MATHEMATICS TEACHER, be given a vote of thanks and complimented for the outstanding work he did during the past six years as editor of this *Journal*. The motion was duly seconded and passed unanimously.

XI. There being no further business, the meeting was duly adjourned at 5:10 P.M.

Respectively submitted,
Houston T. Karnes,
Recording Secretary

Report of the 1959 election of officers

The official count of the results of the election for officers and directors of the Council has been completed by the Remington-Rand Corporation. A report of the count has been transmitted to the president of the Council, to the Chairman of the Committee on Nominations and Elections, to the Executive Secretary, the Board of the National Council, and to the candidates.

The official count shows that the following persons were elected:

Vice-President for College:
Phillip S. Jones

Vice-President for Junior High School:
Mildred B. Cole

Directors:
Philip Peak
Oscar F. Schaaf
Henry Van Engen

A total of 3,747 ballots was cast.

Respectively submitted,
Milton W. Beckman, Chairman,
Committee on Nominations
and Elections

The 1959 budget

The Budget Committee for 1959 was composed of Miss Agnes Herbert, Dr. H. Vernon Price, and Mr. Jackson B. Adkins as Chairman. Due to illness Mr. Adkins had to resign from the Committee. Thereupon, President Harold P. Fawcett appointed Dr. Houston T. Karnes to replace Mr. Adkins.

Immediately following the New York meeting, President Fawcett gave the Committee the following directive:

1. Study the over-all operations of the Central Office wherein budgetary items would be involved.
2. Give consideration to the employment of editorial assistant.
3. Consider an annual audit of the Council's accounts by a Certified Public Accountant.
4. Give consideration to the appointment of a professional assistant.
5. Give consideration to the purchase of further mechanical equipment.
6. Give consideration to defraying the expenses of certain committee meetings and in paying the expenses of certain key people, other than the officers and directors, to the various meetings of the Council.
7. Consider investment of surplus funds.
8. In view of a study of the above items, prepare and submit a budget to be distributed to the Board one month prior to the Dallas meeting.

The Committee met at the Washington office, January 25–27, 1959, and made a thorough study under President Fawcett's directive. During this period, the Committee had the fine services of the Executive Secretary, M. H. Ahrendt, the availability of all the important records in the Central Office, and the opportunity to discuss some of our problems with certain executives of the National Education Association. On the basis of the work during this meeting, a budget was prepared and submitted to the Board on March 1, 1959. At the Dallas meeting, the budget was presented by the Chairman and discussed thoroughly by the Board. The budget as adopted by the Board, which includes only a few changes from the budget as prepared by the Committee, is submitted below:

RECEIPTS		
Memberships (19,549 @ $5.00)	$97,745	
Memberships (Students: 1,400 @ $1.50)	2,100	
MSJ Subscriptions (65,000 @ $0.30)	19,500	
Memberships (for second journal: 4,350 @ $3.00)	13,050	
Subscriptions (both journals: 3,900 @ $7.00)	27,300	
Subscriptions (both journals: 3,900 @ $6.75)	26,325	
Advertising in journals	7,800	
Interest on U.S. Treasury Bonds	375	
Total		$194,195
EXPENDITURES		
Washington Office		
Executive Secretary		
Salary	$10,825	
Special benefits	1,200	
Professional Assistant		
Salary	7,500	
Special benefits	180	
Editorial assistant		
Salary	6,000	
Special benefits	170	

Secretarial and clerical help		
Salaries	50,000	
Special benefits	1,850	
General		
Office expense	14,000	
Travel	3,000	
Total		$ 94,725
President's Office		
Office expense	$ 700	
Travel	2,000	
President's Fund	700	
Secretary	1,820	
Total		$ 5,220
Vice-Presidents		
Postage and secretarial help	$ 2,000	
Total		$ 2,000
THE MATHEMATICS TEACHER		
Printing and mailing	$36,000	
Office expenses and secretary	4,500	
Travel, editor	500	
Total		$ 41,000
The Arithmetic Teacher		
Printing and mailing	$18,000	
Office expenses and secretary	2,000	
Travel, editor	500	
Total		$ 20,500
The Mathematics Student Journal		
Printing and mailing	$12,000	
Office expenses	1,000	
Travel, editor	500	
Total		$ 13,500
Other		
Committee work	$ 5,000	
Travel, directors, vice-presidents, and recording secretary	5,000	
Convention program expenses	1,500	
Audit	750	
Special Projects Fund	5,000	
Total		$ 17,250
TOTAL EXPENDITURES		$194,195

Minutes of the Tenth Delegate Assembly

Baker Hotel, Dallas, Texas

April 1–4, 1959

Edited by Elizabeth Roudebush, Seattle Public Schools, Seattle, Washington

The Tenth Delegate Assembly for Affiliated Groups of The National Council of Teachers of Mathematics met in the Texas Room of the Baker Hotel, Dallas, Texas, at 9:00 A.M., April 2, 1959, with Chairman Elizabeth J. Roudebush presiding.

Dr. Harold P. Fawcett, President of The National Council of Teachers of Mathematics, expressed gratification for the interest in the Council shown by mathematics teachers over the nation. Dr. Fawcett announced an increase in membership of about four thousand over that of last year. He stressed that membership in the National Council carries with it large privileges and opportunities. Though the increase in numbers is gratifying, the primary gain is in the number of teachers over the nation who receive the benefit of the National Council's publications and services. Within a short time, reports from various committees and study commissions from over the nation will be mailed to all members of the Council.

Dr. Fawcett announced the creation of a problem policy committee to examine the role of the National Council in influencing better teaching of mathematics and better training of mathematics teachers. He asked that teachers send suggestions to him or to Howard F. Fehr of what the Council might do more effectively to aid mathematics teachers.

The delegates and alternates were introduced by their respective regional representatives.

An Experiment Concerning Field Service in Mathematics and Science

A panel composed of Wayne Taylor, University of Texas, panel moderator, and John Wagner, University of Texas, discussed "An Experiment Concerning Field Service in Mathematics and Science." Mr. Taylor began by giving a background of the program called "Science Teaching Improvement" which was initiated by the American Association for the Advancement of Science. In implementing this program, four experimental centers were set up to test the use of consultants. One of these programs was established in Texas.

During the time that this program has been in effect, its strength was found to be the personal contact of the consultants with the teachers. The consultants not only aided teachers in methods, but also aided in providing curriculum material, in establishing laboratories and science fairs, and in aiding superintendents and school boards in building planning. The consultants also helped establish mathematics sections in libraries.

Mr. Wagner emphasized that in the beginning the need for new materials and procedures was paramount. Teachers recognized that there must be self-improvement, both as to subject-matter education and competency in presentation.

The consultants are beginning to go to the elementary schools, also; and many educators now realize that the individual consultant program is worth the money to any school system.

How Colleges Are Meeting the Challenge of Training Teachers in Mathematics

The second panel was a discussion of "How Colleges Are Meeting the Challenge of Training Teachers in Mathematics."

Mary Ruth Cook, North Texas State College, Denton, Texas, moderator, said that teachers who are to direct an expanding program in mathematics must be trained. She then introduced the panel members.

Ida Mae Heard, Winnetka, Illinois, talked about training mathematics teachers of grades one through six. Mrs. Heard emphasized:

1. Colleges are carefully selecting the personnel for teacher training. The nature of the teacher training in methods courses is workshop type, and also includes lectures from authoritative people. The teacher training courses try to develop in the students an awareness of available literature.
2. The in-service training consists of workshops and visiting consultants.
3. Curriculum improvement has been most effective in changes of content and placement of topics.

Don Edmondson, Southern Methodist University, Dallas, Texas, said that the training for teachers of junior high is the most unsettled and critical at this time. There are indications that some communication between mathematics and education departments is being established.

W. T. Guy, Jr., University of Texas, Austin, Texas, observed that in the last five years many colleges have improved their teacher training.

He spoke of the Traveling Lecture Program of the Mathematical Association of America, which is to be presented to high school classes as well as to teachers. He mentioned in-service training as well as summer institutes and academic-year institutes.

Max Beberman, University of Illinois, Urbana, Illinois, discussed the Illinois Plan and the teacher training being done at the University of Illinois. He said that this fall the ninth-grade course of the Illinois Plan will be off the press and available for use by any school.

COMMITTEE REPORTS

"Teacher Education, Certification, and Recruitment." Henry Syer, Kent School, Kent, Connecticut, reported that this committee is developing many plans. There is a report on certification available from the A.A.A.S. Commission. This report will be mailed to any member of The National Council of Teachers of Mathematics who writes to Mr. Syer.

"The Elementary School Curriculum." J. Fred Weaver, Boston University, Boston, Massachusetts, reported that THE MATHEMATICS TEACHER will carry a full report from this committee. The committee recommends that the elementary program be tied in with changes in the programs of secondary schools. He asked that NCTM members send any suggestions to him that would be helpful to the committee.

The Mathematics Student Journal. W. Warwick Sawyer, Wesleyan University, Middletown, Connecticut, asked for each regional representative to appoint a representative for the *Journal.* He asked to be informed of how *The Mathematics Student Journal* is received by the students. He hopes to get more two-way correspondence between the *Journal* and the schools.

Membership. Mary C. Rogers, Roosevelt Junior High School, Westfield, New Jersey, gave a very enthusiastic report concerning the membership growth of the National Council of Teachers of Mathematics. The membership, February 1, 1959, was 22,315, with additional members as of April 1.

Miss Roudebush expressed her appreciation to all who participated in the meeting. The meeting of the Tenth Delegate Assembly adjourned at noon.

Respectfully submitted,
Merle Smith,
Norman Bennett,
Secretaries of Delegate Assembly

The following people were chosen by their local groups as delegates and alternates to the Delegate Assembly. Where two names are listed, the first is the delegate, and the second one the alternate.

Mathematics Department of Alabama Education Association
Miss Volena Whaley, 708 Ridgely Apts., Birmingham, Alabama

Arizona Mathematics Association
Mrs. Katharine J. S. Sasse, 1724 E. Miles, Tucson, Arizona
Miss M. Willene Neely, Rinson High School, Tucson, Arizona

Arkansas Council of Teachers of Mathematics
Mrs. Nora Lee Lawless, Box 273 Levy Station, North Little Rock, Arkansas

California Mathematics Council
John Hancock, 3439 Janice Way, Palo Alto, California
Miss Beatrice Truesdale, Santa Barbara High School, Santa Barbara, California.

District of Columbia Teachers of Mathematics
Mrs. Helen N. Cooper, 6105 Swansea St., Bethesda, Maryland
Mrs. Faith F. Novinger, 227 Longfellow St., N.W., Washington 11, D.C.

Benjamin Banneker Mathematics Club
Mrs. Ethel H. Grubbs, 715 Fairmount St., N.W., Washington 1, D.C.

Florida Council of Teachers of Mathematics
Mrs. Jo Anne S. Taber, 1981 Delray Avenue, Jacksonville 10, Florida

Dade County (Florida) Council of Teachers of Mathematics
Robert Gallagher, 3822 S.W. 60th Place, Miami, Florida

Pinellas County Council of Teachers of Mathematics
Robert L. Root, 5702 78th Ave., North Pinellas Park, Pinellas County, Florida
Joe C. Stollmayer, 5116 Gulfport Blvd., St. Petersburg, Florida

Florida State University Mathematics Club
Donald C. Christine, 1623 West Call St., Tallahassee, Florida
Bill Cartwright, 424 East Call St., Tallahassee, Florida

Georgia Council of Teachers of Mathematics
James P. Brown, 2448 Chandler Road, Decatur, Georgia

Illinois Council of Teachers of Mathematics
Miss Mary Louise Fisher, Joliet Township High School & Jr. College, Joliet, Illinois
Miss Mildred Cole, Karl D. Waldo Jr. High School, Aurora, Illinois

Chicago Elementary Teachers Mathematics Club
Joseph J. Urbancek, Chicago Teachers College, Chicago, Illinois

Men's Mathematics Club of Chicago
V. R. Kent, 1510 South 6th Street, Maywood, Illinois
F. A. Janecek, 2423 South Austin Blvd., Cicero, Illinois

Women's Mathematics Club of Chicago and Vicinity
Miss Martha Hildebrandt, East High School, 808 South Second Ave., Maywood, Illinois

Indiana Council of Teachers of Mathematics
Mrs. Eleanor Guyer, Southport High School, 971 Bonta Rd., Indianapolis 27, Indiana

Iowa Association of Mathematics Teachers
Miss Lillian Payne, Mason City High School, Mason City, Iowa

Kansas Association of Teachers of Mathematics
Asel W. Harder, 306 West First, Ellinwood, Kansas

The Wichita Mathematics Association
Donald C. Martinson, 1259 N. Edwards, Wichita 12, Kansas
Mrs. Mary Cowgill, 1625 N. Vassar, Wichita 14, Kansas

Louisiana-Mississippi Branch of National Council of Teachers of Mathematics
Miss Lurnice Begnaud, Lafayette High School, Lafayette, Louisiana
Robert C. Brown, Dept. of Mathematics, Southeastern Louisiana College, Hammond, Louisiana

Mathematics Section of the Maryland State Teachers Association
Miss Margaret A. Bowers, Marriottsville, Maryland

Michigan Council of Teachers of Mathematics
Murel M. Kilpatrick, Lincoln School, Ypsilanti, Michigan

Minnesota Council of Teachers of Mathematics
Harvey Jackson, Minneapolis Public Schools, 807 N.E. Broadway, Minneapolis, Minn.

Missouri Council of Teachers of Mathematics
Mrs. Adeline A. Riefling, 3507 Hawthorne Blvd., St. Louis, Missouri

Nebraska Section—National Council of Teachers of Mathematics
Dr. Milton W. Beckmann, University High School, University of Nebraska, Lincoln, Nebraska
M. Scott Norton, 720 South 22nd, Lincoln, Nebraska

The Association of Teachers of Mathematics in New England
Dr. Henry Syer, Kent, Connecticut

Southern New England Mathematics Association
William G. Shute, Wallingford, Connecticut

New Jersey Mathematics Association
Mr. Gail Koplin, A. L. Johnson Regional High School, Clark, New Jersey
Dr. Max A. Sobel, Montclair State College, Montclair, New Jersey

New York State Association of Mathematics Teachers
Mr. Edward Sherley, Mont Pleasant High School, Schenectady, New York
Mrs. Miriam Howard, Phelps, New York

Association of Teachers of Mathematics of New York City
Dr. Julius Hlavaty, DeWitt Clinton High School, Mosholy Parkway, Bronx 68, New York

Ohio Council of Teachers of Mathematics
Dr. Harold E. Tinnappel, Bowling Green State University, Bowling Green, Ohio

The Mathematics Club of Greater Cincinnati
Miss Sarah Gruenholz, Cincinnati Public Schools, Cincinnati, Ohio

Greater Cleveland Mathematics Club
Lawrence Hyman, Cleveland Public Schools, Cleveland, Ohio

Oklahoma Council of Teachers of Mathematics
Thomas J. Hill, Oklahoma City Public Schools, 900 N. Klein, Oklahoma City 6, Oklahoma
Mrs. Kathryne Leverett, Jackson Jr. High School, Oklahoma City, Oklahoma

Oklahoma City Mathematics Council
Mrs. Virginia Shike, Jackson Jr. High School, Oklahoma City, Oklahoma
Miss Lola Greer, Moon Jr. High School, Oklahoma City, Oklahoma

Tulsa Chapter of National Council of Teachers of Mathematics
Mrs. Bona Gordey, Horace Mann Junior High School, Tulsa, Oklahoma

Oregon Council of Teachers of Mathematics
Dr. Oscar Schaaf, 123 Leigh, Eugene, Oregon
Prof. F. L. Griffin, 3419 East Oregon St., Portland, Oregon

Pennsylvania Council of Teachers of Mathematics
Miss Freda Jones, 372 Schuyler Avenue, Kingston, Pennsylvania

Mathematics Council of Western Pennsylvania
Dr. Catherine A. V. Lyons, 12 South Fremont Ave., Pittsburgh, Pennsylvania

Association of Teachers of Mathematics of Philadelphia and Vicinity
Shaylor Woodruff, 2111 Wharton Rd., Glenside, Pennsylvania

South Carolina Council of Mathematics Teachers
Mrs. Alice R. Rabon, Brookland-Cayce Sr. High School, Cayce, South Carolina

Texas Teachers of Mathematics
Mrs. Mozelle Schulenberger, 101 Ross Avenue, Cleburne, Texas

Alamo District Council of Teachers of Mathematics
Robert C. Zirkel, Highlands High School, 3118 Elgin, San Antonio, Texas
Richard Bennett, 2514 Waverly Avenue, San Antonio, Texas

Greater Dallas Council of Teachers of Mathematics
Dr. Frances Freese, 6856 Casa Loma, Dallas, Texas

Houston Council of Teachers of Mathematics
Mrs. Velma Dickerson, 1918 Hollister, Houston, Texas

Utah Council of Teachers of Mathematics
Miss Eva Crangle, Board of Education, 440 East First South St., Salt Lake City, Utah
Miss Thelma Rickey, East High School, Salt Lake City, Utah

Mathematics Section Virginia Education Association
Mrs. Owen Hoagland, James Ward High School, Winchester, Virginia

Arlington County Association of Teachers of Mathematics
Richard Jameson, Stratford Jr. High School, Arlington, Virginia

Puget Sound Council of Mathematics Teachers
Miss Elizabeth Roudebush, 815 4th Avenue North, Seattle 9, Washington

Wisconsin Mathematics Council
Vincent F. Brunner, Nicolet High School, Milwaukee, Wisconsin

Minutes of the meeting of the Publications Board at Dallas, Texas, April 2 and 4, 1959

The following three motions were passed and submitted to the Board of Directors:

1. The Publications Board considered the question of compensation to authors of NCTM publications, other than journals. This Board went on record as favoring some remuneration for authors, but felt that more time is needed for formulating a specific recommendation on this question. We hope that a definite recommendation can be presented to the Board of Directors at the time of the summer meeting.
2. The Publications Board authorized the printing of 10,000 reprints, with covers, of the Mathematics Curriculum Report of the Secondary School Curriculum Committee.
3. The Publications Board recommends that it meet at least two months prior to the annual Spring Meeting of the NCTM, the time to be arranged by the Chairman, and that expenses be paid according to regular Board policy. This recommendation was approved by the Board of Directors.

Further action at this meeting of the Publications Board was as follows.

4. The suggestion was made at the Christmas Meeting that "certain people of long experience in the NCTM be asked to write an account of important events and outstanding achievements of the Council as they experienced them during their services to the Council, and that these papers, as written, be filed with the Executive Secretary for future reference to significant historical facts." In pursuance of this suggestion, the Chairman of the Publications Board is to prepare a form letter and a list of the appropriate peole to whom it shall be sent, inviting them to write such recollections and send them to the Executive Secretary. Dr. Fawcett will be asked to send the letters out over his signature.
5. With respect to the suggestion that a pamphlet on the history of the NCTM be prepared for the 50th Anniversary of the Council, the Publications Board is in favor of such a pamphlet but thinks it is too early to do anything about it now.
6. The suggestion of Dr. Van Engen in his letter of March 16, 1959, with respect to a pamphlet reviewing the pros and cons of various kinds of grouping and methods to provide for individual differences, especially on the elementary and junior high school level, was referred to the Supplementary Publications Committee for study and recommendation.
7. No action was taken on the suggestion that the NCTM prepare and publish translations of certain Russian and German books suitable for secondary school teachers.
8. The Supplementary Publications Committee was instructed to investigate the possibility of preparing pamphlets for teachers to supplement some of the ideas of the 23rd Yearbook. These pamphlets would provide exercises and suggestions for teaching, and attempt to present some of the material at a simpler level.
9. The Publications Board approved the printing of an additional 5,000 copies of the 23rd Yearbook.
10. It was reported that the first draft of the pamphlet being prepared by Mr. and Mrs. Freitag to replace "Numbers and Numerals" has been returned to the authors for revision, and they hope to work on it this summer. The Publications Board suggests that the title which has been chosen is not suitable and should be changed.
11. A discussion of the possibility of revising "Number Stories of Long Ago" led to the conclusion that at present no revision is necessary.
12. The matter of publishing separately the four articles on introduction to algebra by Dr. Sawyer which have appeared in this year's *Mathematics Student Journal* was referred to the Supplementary Publications Committee with the recommendation that they be published as a pamphlet.
13. As a result of discussion of the suggestion made by Mr. Swain in his letter of March 3, 1959, about a survey of publications dealing with mathematics and the teaching of mathematics which are appearing in various sources, the Chairman of the Publications Board was instructed to recommend to the editors of THE MATHEMATICS TEACHER and the *Arithmetic Teacher* that there be a department in the journals similar to the new department in the journal of the MAA, giving information about such publications.
14. It was agreed that the NCTM should stock

reprints of the new bulletin of the NASSP on New Developments in Mathematics. The number of copies should be a 50% increase over the number of the earlier, similar Bulletin which was sold by the NCTM.

15. A manuscript entitled "Notes on Vectors, Analytics, and Calculus," which was processed earlier by the Supplementary Publications Committee, has since been printed by a local group in New York. The group still would like the NCTM to publish it, and the Publications Board agreed that we should do so.

Our thanks to Henry Swain for this excellent record of the business transacted at the meeting.

Very best wishes to Clifford Bell as he assumes the chairmanship of the committee.

H. Glenn Ayre
Chairman 1958–59

Report of the Membership Committee

Mary C. Rogers, Chairman, Membership Committee,
Edison Junior High School, Westfield, New Jersey

Making our annual membership report for the National Council of Teachers of Mathematics is always a very pleasant assignment, for it enables us to share with you the exceptionally good news which you have helped make possible.

This has been another remarkably successful year in membership growth. Needless to say, we are very happy about it. We are confident that you share this feeling with us, for a great deal of this success is due to you.

Your keen enthusiasm about the growth of the Council and the expansion of its services and your direct assistance as individuals in the realization of this expansion have been major contributing factors toward the growth and increased strength of this great organization. This strong support on your part has been an invaluable aid to the leaders of the Council in the realization of the programs which they are preparing for the improvement of mathematics education at all levels of learning.

Five short years ago, membership in the National Council of Teachers of Mathematics had reached a total of 10,001. Since that time, the membership has been more than doubled; each successive year brings greater and greater current increases to already high totals. When Dr. Howard F. Fehr and the Board of Directors of the Council challenged us in April 1957 to an ultimate 25,000 membership, I think many of us really wondered whether such a goal could be reached. It now seems quite possible that this was too modest a goal.

The official membership count of May 1, 1959 was 23,790. This is only 1210 members short of the 25,000 membership goal originally set for May 1, 1962.

During the past five years, there has been an average annual growth of 2,758 members, but may we suggest that this

average increase is not an adequate picture of membership growth, and of possible future increases in membership totals. You will be interested in the following data:

Increase from May 1954 to May 1955	2723
Increase from May 1955 to May 1956	1103
Increase from May 1956 to May 1957	2354
Increase from May 1957 to May 1958	3316
Increase from May 1958 to May 1959	4293

After the first year's "spurt" in membership growth, records show that each year's increase approximates 1000 greater growth than the increase for the preceding year. May we not anticipate an approximate 5000 membership increase for the current year just beginning? Is it not possible that we can raise the total to nearly 28,500 by May 1960?

Interest in the improvement of mathematics education at all levels is unusually keen at this time. Such interest is not only being expressed by leaders and workers in the field of mathematics education; it is also being voiced by the lay public in our communities, and by the general public, country-wide.

There is an urgency for change in content and in point of view in keeping with newer emphases in mathematics. This urgency—already being felt in our own classrooms—creates for the great majority of mathematics teachers a strong need for expert advice. Let us seize every opportunity to interest these teachers and their administrators in the fine services which the National Council has to offer.

Record of membership growth

In preparing our analyses of membership growth, we are using as bases for comparison the "21,000 goal," the "23,000 goal," and the "25,000 goal." Whether this latter goal will eventually be raised to a still higher goal remains to be seen. What is your opinion of such action?

By far the greater number of states and territories have gone beyond their quotas in the "21,000 goal," but we are including this in our analysis for the encouragement of everyone concerned.

May we call your attention to the following interesting data:

1. The present over-all picture shows we have already reached:
   a. 113% of the "21,000 goal."
   b. 103% of the "23,000 goal."
   c. 85% of the "25,000 goal."
2. Based on the 21,000 goal:
   a. 33 states and 7 of the 10 major territories have gone beyond their quotas.
   b. 11 additional states and the remaining 3 major territories have achieved 90%–99% of such goals.
   c. Only 9 states have not attained such listing.
3. Based on the 23,000 goal:
   a. 24 states and 6 of the 10 major territories have gone beyond their quotas.
   b. 9 additional states and 2 additional major territories have achieved 90%–99% of such goals.
4. Based on the 25,000 goal:
   a. 16 states and 4 of the 10 major territories have gone beyond their quotas.
   b. 10 additional states and 2 additional major territories show 90%–99% achievement.

The following pages present a detailed analysis of National Council membership achievement by states. Please study this for the over-all story it tells and for the more specific data it presents regarding your more immediate interests.

States Relatively Near the 25,000 Goal

	Membership Increase May 1958–May 1959	*Nearness to the 25,000 goal*
Colorado	51	−6
Illinois	297	−170
Kentucky	32	−17
Massachusetts	113	−85
Minnesota	157	−46
Mississippi	20	−30
New Hampshire	12	−2
North Dakota	26	−5
Ohio	127	−134
Oklahoma	48	−20
Pennsylvania	215	−47
Rhode Island	15	−17
Vermont	24	−2
Washington	51	−6
Wisconsin	156	−74

States and Territories Having Reached Their Goals or Gone Beyond Them
(Based on 21,000 goal)

Arizona	331%
Nevada	221%
Oregon	164%
Canada	160%
South Dakota	158%
Idaho	152%
California	143%
Michigan	143%
Utah	143%
Montana	139%
Texas	134%
New York	132%
Florida	127%
Wyoming	126%
New Jersey	122%
Connecticut	121%
New Hampshire	118%
Colorado	117%
Washington	117%
Vermont	116%
Pennsylvania	115%
Oklahoma	113%
Minnesota	111%
North Dakota	111%
Illinois	109%
Kentucky	109%
Ohio	106%
Wisconsin	106%
Massachusetts	105%
Maine	104%
Louisiana	103%
Mississippi	102%
Rhode Island	102%

Membership Achievement 90%–99%
(Based on 21,000 goal)

Maryland	98%
Georgia	95%
Missouri	95%
Hawaii	95%
Foreign	94%
Iowa	94%
Kansas	94%
Dist. of Columbia	92%
New Mexico	92%
Virginia	91%
Alabama	90%

States and Territories Having Reached Their Goals or Gone Beyond Them
(Based on 23,000 goal)

Arizona	302%
Nevada	200%
Oregon	149%
Canada	146%
South Dakota	144%
Idaho	136%
California	131%
Michigan	130%
Utah	130%
Montana	127%
Texas	122%
New York	121%
Florida	116%
Wyoming	115%
New Jersey	112%
Connecticut	110%
Colorado	107%
New Hampshire	107%
Washington	107%
Vermont	106%
Pennsylvania	105%
Oklahoma	104%
Minnesota	101%
North Dakota	101%

Membership Achievement 90%–99%
(Based on 23,000 goal)

Illinois	99%
Kentucky	99%
Ohio	97%
Wisconsin	97%
Massachusetts	96%
Maine	95%
Mississippi	93%
Rhode Island	93%
Louisiana	91%

States and Territories Having Reached Their Goals or Gone Beyond Them
(Based on 25,000 goal)

Arizona	278%
Nevada	183%
Oregon	137%
Canada	135%
South Dakota	132%
Idaho	127%
California	120%
Michigan	120%
Utah	120%
Montana	117%
New York	112%
Texas	112%
Florida	107%
Wyoming	105%
New Jersey	103%
Connecticut	102%

Membership Achievement 85%–99%
(Based on 25,000 goal)

Colorado	98%
New Hampshire	98%
Washington	98%
Pennsylvania	97%
Vermont	97%
Oklahoma	95%
Minnesota	93%
North Dakota	93%
Kentucky	92%
Illinois	91%
Ohio	89%
Wisconsin	89%
Maine	88%
Massachusetts	88%
Louisiana	87%
Rhode Island	86%
Mississippi	85%

Leaders in Membership Totals (Not including subscriptions)	
New York	1518
Illinois	1272
Pennsylvania	1067
California	999
Texas	883
Ohio	829
Michigan	799
New Jersey	661
Indiana	491
Massachusetts	472
Florida	441
Wisconsin	438
Minnesota	427
Virginia	343
Kansas	322

Leaders in Membership Totals (Including subscriptions)	
New York	2184
Illinois	1810
California	1765
Pennsylvania	1486
Texas	1260
Michigan	1200
Ohio	1099
New Jersey	961
Massachusetts	648
Wisconsin	626
Florida	624
Indiana	623
Minnesota	622
Virginia	456
Canada	451

Membership Analysis by Territory

	May 1, 1958			*May 1, 1959*			
		Per Cents of Goals Reached			*Per Cents of Goals Reached*		
	Membership	*19,000*	*21,000*	*Membership*	*21,000*	*23,000*	*25,000*
Pearl Bond Alabama, Louisiana, Mississippi, Texas	1,578	100%	91%	2,047	118%	108%	99%
Marian Cliffe Arizona, California, New Mexico, Utah	1,819	134%	121%	2,253	150%	137%	126%
Mary Lee Foster Arkansas, Kentucky, Missouri, Nevada, Tennessee	937	84%	76%	1,237	92%	85%	78%
Janet Height New England States	1,190	103%	93%	1,408	110%	101%	93%
Lucille Houston Michigan, Minnesota, Ontario, South Dakota, Wisconsin	2,210	105%	95%	2,321	128%	117%	108%
Harold J. Hunt Hawaii, Idaho, Montana, North Dakota, Oregon, Washington	756	124%	112%	968	124%	113%	104%
Florence Ingham Colorado, Kansas, Nebraska, Oklahoma, Wyoming	1,164	99%	90%	1,331	102%	93%	86%
Mildred Keiffer Illinois, Indiana, Iowa, Ohio, West Virginia	3,482	94%	85%	4,056	99%	91%	83%
Faith Novinger Delaware, Dist. of Columbia, New Jersey, New York, Pennsylvania	3,993	108%	98%	4,995	121%	111%	102%
Bess Patton Florida, Georgia, Maryland, North Carolina, South Carolina, Virginia	1,885	95%	86%	2,152	98%	89%	82%

The National Council of Teachers of Mathematics Analysis of Membership Growth, Members, and Subscribers, May 1958 to 1959

	May 1958	May 1959		Goals and Per Cents of Goals Reached					
		Individuals	Totals	21,000		23,000		25,000	
				Goal	%	Goal	%	Goal	%
Alabama	177	154	214	237	90%	260	82%	283	76%
Alaska		12	17						
Arizona	203	198	278	84	331%	92	302%	100	278%
Arkansas	170	131	197	239	82%	262	75%	285	69%
California	1,445	999	1,765	1,231	143%	1,348	131%	1,465	120%
Colorado	243	233	294	252	117%	276	107%	300	98%
Connecticut	339	268	386	319	121%	350	110%	380	102%
Delaware	82	67	87	99	88%	108	81%	118	74%
Dist. of Columbia	211	188	237	258	92%	283	84%	308	77%
Florida	528	441	624	491	127%	538	116%	585	107%
Georgia	226	174	248	260	95%	285	87%	310	80%
Idaho	27	27	38	25	152%	28	136%	30	127%
Illinois	1,513	1,272	1,810	1,663	109%	1,822	99%	1,980	91%
Indiana	544	491	623	758	82%	830	76%	900	69%
Iowa	338	259	391	414	94%	453	86%	493	79%
Kansas	377	322	411	435	94%	476	86%	518	79%
Kentucky	156	135	188	172	109%	189	99%	205	92%
Louisiana	356	254	398	386	103%	423	91%	460	87%
Maine	94	83	101	97	104%	106	95%	115	88%
Maryland	365	265	390	399	98%	437	89%	475	82%
Massachusetts	535	472	648	615	105%	674	96%	733	88%
Michigan	900	799	1,200	840	143%	920	130%	1,000	120%
Minnesota	465	427	622	561	111%	614	101%	668	93%
Mississippi	155	123	175	172	102%	189	93%	205	85%
Missouri	350	321	438	462	95%	506	87%	550	80%
Montana	97	89	117	84	139%	92	127%	100	117%
Nebraska	164	136	188	231	81%	253	74%	275	68%
Nevada	25	33	42	19	221%	21	200%	23	183%
New Hampshire	94	79	106	90	118%	99	107%	108	98%
New Jersey	743	661	961	785	122%	860	112%	935	103%
New Mexico	101	69	114	124	92%	136	84%	148	77%
New York	1,686	1,518	2,184	1,653	132%	1,810	121%	1,958	112%
North Carolina	237	185	274	357	77%	391	70%	425	64%
North Dakota	42	46	68	61	111%	67	101%	73	93%
Ohio	972	829	1,099	1,035	106%	1,134	97%	1,233	89%
Oklahoma	327	277	375	332	113%	359	104%	395	95%
Oregon	267	191	306	187	164%	205	149%	223	137%
Pennsylvania	1,271	1,067	1,486	1,287	115%	1,410	105%	1,533	97%
Rhode Island	86	77	101	99	102%	108	93%	118	86%
South Carolina	127	99	160	191	84%	209	77%	228	70%
South Dakota	54	45	79	50	158%	55	144%	60	132%
Tennessee	236	203	281	344	82%	377	75%	410	69%
Texas	890	883	1,260	941	134%	1,030	122%	1,120	112%
Utah	70	56	96	67	143%	74	130%	80	120%
Vermont	42	53	66	57	116%	62	106%	68	97%
Virginia	402	343	456	500	91%	547	83%	595	77%
Washington	323	220	374	319	117%	350	107%	380	98%
West Virginia	115	91	133	220	60%	242	55%	263	51%
Wisconsin	470	438	626	588	106%	644	97%	700	89%
Wyoming	53	41	63	50	126%	55	115%	60	105%
TOTALS	18,693	15,844	22,795	20,140	113%	22,059	103%	23,977	95%
Hawaii & U. S. Poss.	92	59	98	103	95%	113	87%	123	80%
Canada	321	208	451	282	160%	308	146%	335	135%
Foreign	391	201	446	475	94%	520	86%	565	79%
GRAND TOTAL	19,497	16,312	23,790	21,000	113%	23,000	103%	25,000	95%

The National Council of Teachers of Mathematics Analysis of Membership Growth, May 1958 to May 1959

	May 1958	*May 1959*	*Increase*	*% Increase*	*Comparison with 25,000 goal*
Alabama	177	214	37	21%	−69
Alaska		17	17		+17
Arizona	203	278	75	37%	+178
Arkansas	170	197	27	16%	−88
California	1,445	1,765	320	22%	+300
Colorado	243	294	51	21%	−6
Connecticut	339	386	47	14%	+6
Delaware	82	87	5	6%	−31
Dist. of Columbia	211	237	26	12%	−71
Florida	528	624	96	18%	+39
Georgia	226	248	22	10%	−62
Idaho	27	38	11	41%	+8
Illinois	1,513	1,810	297	20%	−170
Indiana	544	623	79	15%	−277
Iowa	338	391	53	16%	−102
Kansas	377	411	34	9%	−107
Kentucky	156	188	32	21%	−17
Louisiana	356	398	42	12%	−62
Maine	94	101	7	7%	−14
Maryland	365	390	25	7%	−85
Massachusetts	535	648	113	21%	−85
Michigan	900	1,200	300	33%	+200
Minnesota	465	622	157	34%	−46
Mississippi	155	175	20	13%	−30
Missouri	350	438	88	25%	−112
Montana	97	117	20	21%	+17
Nebraska	164	188	24	15%	−87
Nevada	25	42	17	68%	+19
New Hampshire	94	106	12	13%	−2
New Jersey	743	961	218	29%	+26
New Mexico	101	114	13	13%	−34
New York	1,686	2,184	498	30%	+226
North Carolina	237	274	37	16%	−151
North Dakota	42	68	26	62%	−5
Ohio	972	1,099	127	13%	−134
Oklahoma	327	375	48	15%	−20
Oregon	267	306	39	15%	+83
Pennsylvania	1,271	1,486	215	17%	−47
Rhode Island	86	101	15	17%	−17
South Carolina	127	160	33	26%	−68
South Dakota	54	79	25	46%	+19
Tennessee	236	281	45	19%	−129
Texas	890	1,260	370	42%	+140
Utah	70	96	26	37%	+16
Vermont	42	66	24	57%	−2
Virginia	402	456	54	13%	−139
Washington	323	374	51	16%	−6
West Virginia	115	133	18	16%	−130
Wisconsin	470	626	156	33%	−74
Wyoming	53	63	10	19%	+3
TOTAL	18,693	22,795	4,102	22%	−1,182
Hawaii & U. S. Poss.	92	98	6	7%	−25
Canada	321	451	130	39%	+116
Foreign	391	446	55	14%	−119
GRAND TOTAL	19,497	23,790	4,293	22%	−1,210

LEADERS IN MEMBERSHIP GROWTH
(Including subscriptions)

New York	498
Texas	370
California	320
Michigan	300
Illinois	297
New Jersey	218
Pennsylvania	215
Minnesota	157
Wisconsin	156
Canada	130
Ohio	127
Massachusetts	113
Florida	96
Missouri	88
Indiana	79

GREATEST PER CENT OF GROWTH
(Compared with May 1958 totals)

Nevada	Utah
North Dakota	Minnesota
Vermont	Michigan
South Dakota	Wisconsin
Texas	New York
Idaho	New Jersey
Canada	South Carolina
Arizona	

STATES AND TERRITORIES
WITH CONTINUOUS GROWTH
(Since May 1958)

Alabama	Nevada
California	New Hampshire
Canada	New Jersey
Colorado	New Mexico
Connecticut	New York
Dist. of Columbia	Oregon
Hawaii	Rhode Island
Illinois	South Dakota
Michigan	Tennessee
Minnesota	Texas
Missouri	Wisconsin
Nebraska	

FUTURE PLANS

1. We are well aware of the current emphasis being laid on the improvement of mathematics education at all levels of learning. Many agencies are at work to bring about the desired improvements. Rapid strides are being made toward the adoption of newer and better programs of instruction. Let us "step up" our publicity and publicity releases in light of these changes. In so doing, let us acquaint teachers of mathematics and their administrators with the fine services of the National Council of Teachers of Mathematics in this transition period.
   a. We cannot emphasize too strongly the importance to the classroom teacher of the excellent and timely articles in THE MATHEMATICS TEACHER and *The Arithmetic Teacher*. Most stimulating to the mathematics student are the current issues of *The Mathematics Student Journal*.
   b. Attention should also be called to the Council yearbooks and to the special prices being given Council members on these yearbooks. Currently invaluable are the Twenty-third Yearbook, *Insights into Modern Mathematics;* and the twenty-fourth Yearbook, *the Growth of Mathematical Ideas, Grades K-12*.
   c. Of great value also are the many supplementary publications and the publication of results of research in mathematics education by the National Council.
   d. Equally valuable to teachers of mathematics will be the recommendations of the National Council Curriculum Committees.
   e. Many other educational agencies have published or are planning early publication of the results of their research and experimentation. Very valuable recommendations will be included in these publications, together with many complete instructional units. These publications will be available to National Council members free of charge or at an appreciably reduced price.
   f. Teachers of mathematics should be made acquainted with the excellent programs of National Council conventions and should be stimulated to attend as many of these meetings as possible.

2. At this time, particularly, you who are members of the mathematics staff in our state colleges and other great schools of education can be of tremendous help in securing members for the National Council. This applies to undergraduate students—potential junior members of National Council—as well as the many graduate students, most of whom are employed in the field of mathematics education or in the general education of the elementary school. Students qualify for junior membership throughout the four years of their undergraduate study, provided their status as students is endorsed each year with the National Council office in Washington by the school of education which they attend. It should be noted that the "student membership fee remains the same as formerly, with one journal available for $1.50 a year, and both for $2.50."
3. Directors and staff members at mathematics institutes; leaders of mathematics work-study groups; supervisors and/or mathematics department chairmen—all are performing an increasingly fine service in encouraging membership in the National Council. We are most grateful to you for this assistance.
4. In their frequent references to National Council publications and/or other services, participants on mathematics programs—local, state, and national—are doing much to stimulate interest in the Council.
5. State and local associations of mathematics teachers, in co-operation with state representatives to the National Council, are performing an outstanding service to the Council. A very great deal of the strength of the Council is due to the efficient and diligent work of the State Representatives supported by officers and members of the Affiliated Groups. The Council is deeply appreciative to you for your support. A continuance of your fine support is vital to the success of Council services.
6. We recommend that library and other institutional subscriptions continue to be included in preparing reports of membership growth. We feel that many teachers of mathematics are reached by these subscriptions—teachers young in experience, with meager incomes—who otherwise would not feel they could afford the services of the Council.
7. We are very much pleased to note the publication of the current National Council directory. We urge persons working to obtain memberships and to call teachers' attention to the importance of being listed in this directory. Such listing seldom fails to please administration.

The Membership Committee extends to each of you its sincere thanks for your outstanding co-operation and helpfulness. Please accept our best wishes for your greatest professional success and growth at all times.

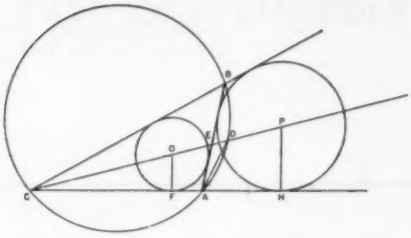
Pearl Bond
Marian C. Cliffe
Mary Lee Foster
Janet Height
Harold J. Hunt
Lucille Houston
Florence Ingham
Mildred Keiffer
Faith Novinger
Bess Patton

M. H. Ahrendt and
Elizabeth Roudebush,
Ex Officio members

Mary C. Rogers, Chairman

Letters to the editor

The three letters printed here were written in response to Prof. John P. Hoyt's letter that appeared on page 388 of the May 1959 issue of THE MATHEMATICS TEACHER. The figure is reproduced for the second time and is the one he provided.—*Editor.*



Dear Editor:

My colleague, Mr. Edward Darke, brought to my attention Prof. Hoyt's letter in the May 1959 issue of THE MATHEMATICS TEACHER. He was amazed (and I am, too) that Prof. Hoyt used such a long and involved proof for a problem set for secondary school students.

The following proof was suggested by Mr. Darke, and I here reproduce it with his permission.

Theorem: The midpoint of the line joining the center of the circle inscribed in a triangle to the center of one of the escribed circles of the triangle lies on the circumscribed circle of the triangle.

In the figure, O is the center of the circle inscribed in $\triangle ABC$; P is the center of the escribed circle touching CA produced, CB produced, and AB; D is the midpoint of OP.

$\angle OAP = 90°$ (OA, AP internal and external bisectors respectively of $\angle BAC$).

Similarly, $\angle OBP = 90°$.

$\therefore$ OP is the diameter of a circle passing through A, O, B, P ($\angle$s in semicircle—converse).

$\therefore$ $\angle ODA = 2\angle OBA$ ($\angle$ at center).

But, $2\angle OBA = \angle CBA$ (OB bisects $\angle CBA$).

$\therefore$ $\angle CDA = \angle CBA$.

$\therefore$ D is a point on the circumcircle of $\triangle ABC$ ($\angle$s in the same segment, are equal—converse).

Incidentally, I used a slightly longer proof in which I showed that $\angle BDA + \angle BCA = 180°$.

Yours sincerely,
IAN ISAACS
Knox College
Spaldings, Jamaica, B.W.I.

Dear Editor:

As a new high school teacher of mathematics, I was fascinated by Mr. John Hoyt's solution of the geometry problem which he presented in his letter to the editor appearing in the May 1959 issue of THE MATHEMATICS TEACHER. It occurred to me, however, that since the problem had been selected for use in a mathematics contest, there must be a more elementary approach to solving it. After borrowing Mr. Hoyt's figure and some of his statements, I was able to arrive at the following solution which I would like to submit in outline form.

Theorem: The midpoint of the line joining the center of the circle inscribed in a triangle to the center of the escribed circles of the triangle lies on the circumscribed circle of the triangle.

Let circles O, P and Q be the inscribed, escribed, and circumscribed circles, respectively, with circles O and P both being tangent to side AB of triangle ABC.

Let V, R, and F be the points of tangency of the inscribed circle to the sides CB, AB, and AC, respectively. Let U, T, and H be the points of tangency of the escribed circle to these same sides, respectively.

Let the circumscribed circle Q intersect OP at D.

PO produced passes through C, since O and P together determine the bisector of angle ABC.

Draw TP, QD, and OR. Being radii drawn to points of tangency, TP is perpendicular to AB, and OR is perpendicular to AB.

Since CP is bisector of angle ABC, arc AD equals arc BD. Therefore, QD is the perpendicular bisector of AB at S.

(1) $VU = FH$. Tangents to same two circles.
(2) $BU = BT$; $VB = BR = BT + TR$. Tangents drawn from same point.
(3) $VU = VB + BU = BR + BU = 2BT + TR$. Substitution.
(4) $AF = AR$; $AH = AT = AR + TR$. Tangents drawn from same point.
(5) $FH = AH + AF = 2AR + TR$. Substitution.
(6) $2AR + TR = 2TB + TR$. Steps (3) and (5).
(7) $AR = BT$. Subtract TR in step (6) and divide remainder by 2.
(8) $AS = BS$. QD is perpendicular bisector of AB.
(9) $AS - AR = BS - BT$. From steps (7) and (8), or $RS = ST$.
(10) Since $TP \perp AB$, $QD \perp AB$ and $OR \perp AB$, $OR \parallel QD \parallel TP$.
(11) Hence parallel lines TP, QD, and OR cut off equal segments TS and SR on the transversal AB. Therefore, segments OD and DP on transversal OP are also equal.

Sincerely yours,
NED HARRELL
Commander, U. S. Navy (Ret.)
Menlo-Atherton High School
Atherton, California

Dear Editor:

On page 388 of the May 1959 issue of THE MATHEMATICS TEACHER there is an interesting algebraic proof by Prof. Hoyt of a theorem that had been posed as a problem in the Polish mathematical Olympiad.

A very simple proof is to be found on pages 68–69, section 102, of Altschiller-Court's *College Geometry* (1925). Essentially, the theorem in question exhibits one of the many properties of the nine-point circle of a triangle.

For, since the interior and exterior angle bisectors issuing from the same vertex are perpendicular to each other, the exterior angle bisectors of the original triangle form the sides of a new triangle having as altitudes the interior angle bisectors of the first triangle. The circumcircle of the original triangle is then easily identifiable as the nine-point circle of the new triangle. The incenter of the first triangle is the orthocenter of the second; the vertices of the first triangle are the feet of the altitude of the other. Hence, the line joining the centers of the inscribed and escribed circles of the first triangle is the segment of the altitude of the second triangle between orthocenter and vertex. The nine-point circle passes through the midpoints of these segments, proving the present theorem.

Very truly yours,
DANIEL MALAMENT
Brooklyn 10, New York

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

JOINT MEETING WITH MATHEMATICAL ASSOCIATION OF AMERICA

January 30, 1960
Conrad-Hilton Hotel, Chicago, Illinois
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

THIRTY-EIGHTH ANNUAL MEETING

April 20–23, 1960
Statler-Hilton Hotel, Buffalo, New York
Louis F. Scholl, Board of Education, Buffalo 2, New York

JOINT MEETING WITH NEA

June 29, 1960
Los Angeles, California
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

TWENTIETH SUMMER MEETING

August 21–24, 1960
University of Utah, Salt Lake City, Utah
Eva A. Crangle, Board of Education, Salt Lake City 11, Utah

Other professional dates

Women's Mathematics Club of Chicago and Vicinity

November 7, 1959
Stouffer's Restaurant, Randolph and Michigan, Chicago, Illinois
Sarane Starr, University of Chicago High School, Blaine Hall, Chicago 37, Illinois

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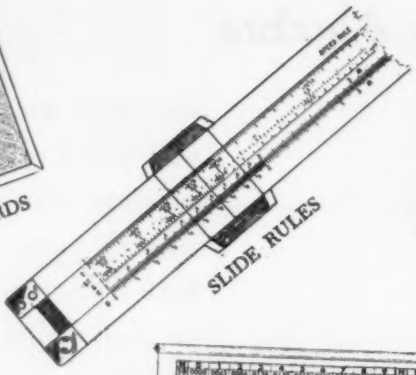

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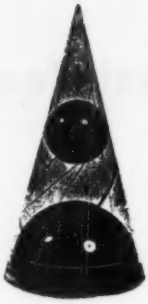

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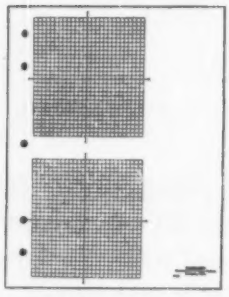

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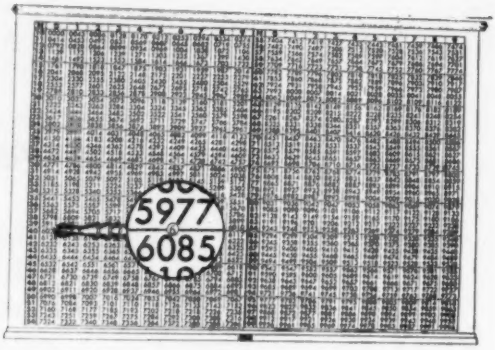

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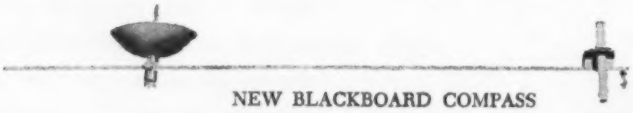

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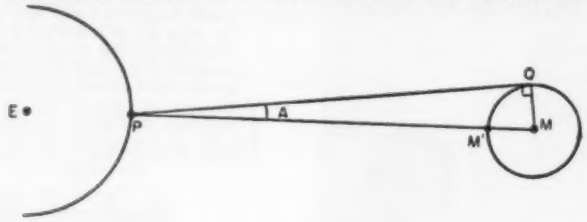

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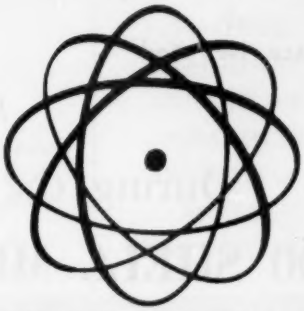

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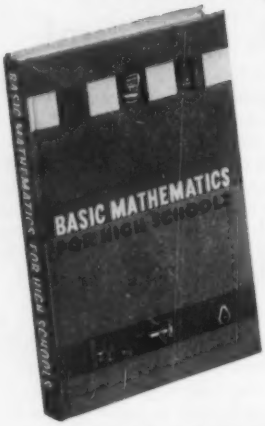

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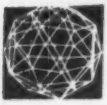

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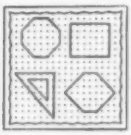

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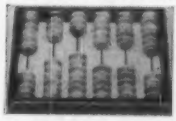

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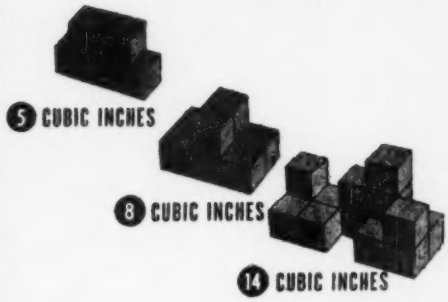

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